2015

Graph determined symbolic dynamics and hybrid systems

Kimberly Danielle Ayers
Iowa State University

Graph determined symbolic dynamics and hybrid systems

by

Kimberly Danielle Ayers

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:

Wolfgang Kliemann, Major Professor

Justin Peters

Paul Sacks

Eric Weber

Stephen Willson

Iowa State University

Ames, Iowa

2015

DEDICATION

I would like to dedicate this thesis to my parents Jon and Helaine, as well as my sisters Lauren and Margaret, without whose support I would not have been able to complete this work. I would also like to thank my friends and family for their loving guidance and assistance during the writing of this work.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this thesis. First and foremost, Dr. Wolfgang Kliemann for his, patience and support throughout this research and the writing of this thesis. His work and insight have truly inspired me in my study of mathematics and life.

I would also like to thank my committee members for their efforts and contributions to this work: Dr. Justin Peters, Dr, Paul Sacks, Dr. Eric Weber, and Dr. Stephen Willson. In addition to mentoring me through my research, they have provided endless support through my classes and overall graduate career. They have influenced me through their work as well as their teaching style.

Finally, I'd like to thank my fellow graduate students in the Iowa State Mathematics Department,for all of their support, both academically and emotionally, throughout my graduate career.

ABSTRACT

In this paper we explore the concept of symbolic dynamical systems whose structure is determined by a directed graph, and then discrete-continuous hybrid systems that arise from such dynamical systems. Typically, symbolic dynamics involve the study of a left shift of a bi-infinite sequence. We examine the case when the bi-infinite system is dictated by a graph; that is, the sequence is a bi-infinite path of a directed graph. We then use the concept to study a system of dynamical systems all on the same compact space M, where "switching" between the systems occurs as given by the bi-infinite sequence in question. The concepts of limit sets, chain recurrent sets, chaos, and Morse sets for these systems are explored.

CHAPTER 1. INTRODUCTION

Symbolic dynamics is the practice of examining a dynamical system that consists of a discrete space made up of infinite or bi-infinite (infinite forwards and backwards) sequences and a shift operator. It helps to think of these sequences as functions from either the integers or natural numbers onto a set of abstract symbols. Thus, not only does the order of the symbols in the sequence distinguish two different sequences, but also how they are indexed - for bi-infinite sequences, one can think of this as where 0 maps to, or where the "origin" point is. Past work in symbolic dynamics have included studying bi-infinite sequences whose entries are made up of only 0s and 1s. This system was initially used to prove the existence of chaotic behavior, first proposed by Henri Poincaré in the 1880s when he discovered the existance of aperiodic orbits that are contained in a bounded set while not approaching a fixed point. This was formalized by Stephen Smale in (9) with his construction of what is known today as the Smale Horseshoe. It can be shown that the dynamics given by the Smale Horseshoe are topologically conjugate to the symbolic dynamical system described above.

In this paper, we examine a symbolic dynamical system that is made up of bi-infinite sequences that are constructed from a directed graph. That is, given a directed graph $G = (V, E)$ where V is the set of vertices and $E \subset (V \times V)$, we consider the set of functions

$$f : \mathbb{Z} \to V$$

where $(f(i), f(i+1)) \in E$ for all $i \in \mathbb{Z}$. We name this set Ω. In this paper, rather than thinking of the elements of Ω as functions, we refer to them as bi-infinite sequences. This makes visualizing the work that is done easier, but using the definition of functions is a clear way of introducing the concept. The shift operator Φ is given by

$$\Phi(f(i), t) = f(i + t)$$

where $t \in \mathbb{Z}$ is the time associated with the flow. In order to study concepts such as limit sets, we require a topology on Ω. We propose a metric on Ω and are thus able to study limit and recurrence behavior. We begin by showing that Ω is compact, and thus ω, α-limit sets are non-empty. We also show that the shift operator is continuous (in the f component) with respect to this metric and characterize the locations of the ω, α-limit sets. We identify Poincaré sets, nonwandering sets, and chain recurrent sets, and end by characterizing a finest Morse decomposition on the system, showing that behavior within the Morse sets is (except for a simple exception) chaotic, just as in seen in the original symbolic dynamical systems, despite the restriction on the order of the sequences that appear in Ω.

A hybrid dynamical system is defined as one that exhibits both continuous and discrete behavior. The canonical example of a hybrid dynamical system is a bouncing ball; while the ball is in the air, its position, velocity, etc are all changing smoothly and continuously in time. However, the moment the ball hits the ground, it instantaneously (discretely) changes direction and moves upward. In the second part of this paper, we examine a hybrid system of the following form: given a finite number of continuous time flows $\phi_1, \phi_2, \ldots, \phi_k$ on the same compact space $M \subset \mathbb{R}^n$, what happens if we switch discretely between the flows at regular time intervals, especially if the switching between the systems is restricted by a directed graph? That is, given a directed graph G on k vertices, each corresponding to some ϕ_i, what does limit behavior look like if the only switching that is allowed is between flows for which there is an edge connecting their associated vertices in the appropriate direction?

In order to study this, we first adapt results from the previous section on symbolic dynamics to fit within the context of continuous time, and then create a skew-product flow on $M \times \Delta$, where Δ is a kind of "continuization" of Ω (explained more rigorously in Chapter 4). In (4), Colonius and Kliemann examine these questions for a system $M \times U$, where U is the set of L^1 functions on $\mathbb{R}$. We consider an adaptation of the concept of chain recurrence in this concept for the space M (since M paired with the systems $\phi_1, \phi_2, \ldots, \phi_k$ is not in its own right a dynamical system, we can not consider chain recurrence as it is defined in the literature on this space). We examine how these so-named "chain sets" are related chain transitive sets that exist in $M \times \Delta$ via projections and lifts. In the case examined in (4), it is shown that chain sets in M

and chain transitive sets in $M \times U$ are exactly projections and lifts of one another. However, it turns out that this relationship is not so simple in the case of $M \times \Delta$; it's highly dependent on the structure of the underlying graph G that determines which switches are allowed. We conclude with a discussion of some examples, and suggest some future directions for this work.

CHAPTER 2. BACKGROUND INFORMATION

We begin by introducing some already established definitions and theorems on directed graphs and dynamical systems. The majority of the definitions, theorems, and notation in this section were borrowed from (1). The proof of all of these theorems can be found there as well. We introduce basic definitions for directed graphs associated with symbolic dynamical systems that will be used in the remainder of the paper. It should be noted that the concepts and theorems presented in this chapter are well known results and constructs within the fields of graph theory and dynamical systems, and do not comprise original work. It should also be noted that some of these definitions used are a bit different than those used in the field of graph theory.

2.1 Directed Graphs

A graph is a discrete structure comprised of vertices and edges, which connect the vertices. Much of graph theory works with simple graphs - those graphs for which the edges do not have associated directions, there is at most one edge between two vertices, and no edge may connected a vertex to itself. However, we will be working with directed graphs, defined next.

Definition 1. *A finite directed graph $G = (V, E)$ is a pair of sets $V = \{1, ..., n\}$ called vertices, and $E \subseteq V \times V$, called edges.*

Definition 2. *Any graph $G = (V, E)$ has a set $\mathcal{P} = \{(x_1, ..., x_k), |(x_i, x_{i+1}) \in E\ ;\ i, k \in \mathbb{N}\}$. Any element of $\mathcal{P}$ is called a path of G with length $k - 1$.*

This definition of edges as a subset of the cartesian product of the set of vertices gives these edges a direction; for instance, the edge $(1, 2)$ is the edge directed away from vertex 1 and towards vertex 2, while the edge $(2, 1)$ is the edge directed away from vertex 2 and towards vertex 1. This also allows for what in standard graph theory is known as "loops", which we here will call a self-edge (the word loop has another meaning for us). These are edges that go from one vertex back to the same vertex, as in the edge $(1, 1)$. In addition, there may now be up to two edges between any two given vertices, one in each direction.

Definition 3. *Any path that begins and ends at the same vertex, regardless of its length, will be called a loop.*

Definition 4. *The out-degree of any $\alpha \in V$, denoted $o(\alpha)$, is the number of $\gamma \in \mathcal{P}$ with length 1 and $\gamma_1 = \alpha$ (here $\gamma = (\gamma_1, \ldots, \gamma_f)$).*

Definition 5. *The in-degree of any $\alpha \in V$, denoted $i(\alpha)$, is the number of $\gamma \in \mathcal{P}$ with length 1 and $\gamma_F = \alpha$.*

Put simply, the out-degree of a vertex is the number of edges that leave that vertex, and the in-degree is the number of edges entering that vertex.

In order for us to consider symbolic dynamics on graphs, it is necessary that we be able to construct paths of arbitrarily long finite length - that is, every path must be able to be extended. This requires a certain type of graph, defined next.

Definition 6. *A finite directed graph $G = (V, E)$ with $o(\alpha) \geq 1, i(\alpha) \geq 1$; for all $\alpha \in V$ is called an N-graph.*

There are particular structures that exist within these N-graphs that have particularly "nice" properties. As seen later, these structures are of incredible importance.

Definition 7. *A communicating class in an N-graph $G = (V, E)$ is a subset $C \subseteq V$ for which two things are true:*

1. *For all $\alpha, \beta \in C$ there exists $\gamma \in \mathcal{P}$ such that $\gamma_1 = \alpha$ and $\gamma_F = \beta$.*

2. *There exists no $C' \supset C$ where for all $\alpha', \beta' \in C'$ there exists $\gamma' \in \mathcal{P}$ such that $\gamma'_1 = \alpha'$ and $\gamma'_F = \beta'$. This condition is called maximality.*

In the literature, communicating classes are also known as maximal strongly connected subgraphs. The idea is that every vertex in a communicating class is able to "communicate" with any other vertex in the communicating class. Thus, we do not require our N-graphs themselves to be strongly connected; in fact they may even not be weakly connected, although this then breaks the study of the system down into the parts created by the connected components of the graph. However, we will see later that every N-graph must contain at least one communicating class.

It is important to note that, in the above definition, the empty set is not a communicating class because the empty set does not meet the requirements for maximality.

Definition 8. *Communicating classes can be classified further in two ways:*

1. *A communicating class C is variant if there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in C$ and $\gamma_F \notin C$.*
2. *A communicating class C is invariant if for all $\gamma \in \mathcal{P}$ with $\gamma_1 \in C$, $\gamma_F \in C$.*

In the future, it will be important for us to have some distinguishing identifying properties of vertices that are contained in communicating class. This next lemma gives an example of such a property.

Lemma 9. *If a vertex x is contained in a loop, then it is contained in a communicating class.*

Proof. Let x be a vertex in G such that there exists a path $P = \{x = x_0, x_1, \ldots, x_n = x\}$. Then notice for all $x_i, x_j \in P$, there exists a path from x_i to x_j by, if $i < j$, taking the path $\{x_i, x_{i+1}, \ldots, x_j\}$, and if $i \geq j$, taking the path $\{x_i, x_{i+1}, \ldots, x, x_1, \ldots, x_j\}$. Thus, the set $\{x, x_1, \ldots, x_{n-1}\}$ satisfies condition 1 of being a communicating class, and thus at the very least is contained in a communicating class, if it itself is not maximal. □

Finally, as mentioned above, in order to consider paths of arbitrarily long length, we must first show that such paths exist in N-graphs.

Theorem 10. *Every N-graph has a path of arbitrarily long finite length and contains a communicating class.*

Proof. (by induction) Let G be an N-graph. Since $o(\alpha) \geq 1$ for all $\alpha \in V$, any $\alpha \in V$ must have associated at least one $e = (\alpha, \eta) \in E$ which constitutes a path of length 1 in G. Assume there exists a path $\gamma \in \mathcal{P}$ of length k from $\gamma_1 = \alpha \in V$ to $\gamma_{k+1} = \beta \in V$. Since $o(\gamma_{k+1} = \beta) \geq 1$, there exists a $\alpha' \in V$ with $e' = (\gamma_{k+1}, \alpha') \in E$. Therefore, there exists a path of length k+1 from γ_1 to α'. Therefore, G contains some path of arbitrarily large finite length.

Next, consider an N-graph G with n vertices. From above, we know there exists a path γ length n. Hence, this path contains $n + 1$ vertices. So for at least 1 pair $1 \leq i < j \leq n + 1$, we have $\gamma_i = \gamma_j$. Therefore, there exists a loop from γ_i to γ_j, which by Lemma 9 implies that g_i is contained within a communicating class, thus proving the existence of a communicating class. □

This theorem then leads to the useful corollary that one can reach at least one invariant communicating class from any vertex in G. This corollary will be proven below, but first we require a little bit of equipment.

Definition 11. *Given a vertex $\alpha \in V$,*

$$O^+(\alpha) = \{x \in V | \text{ there exists a path from } \alpha \text{ to } x\}$$

is the positive orbit of α. The negative orbit of α, $O^-(\alpha)$, is defined similarly.

Lemma 12. *Let $\alpha, \beta \in V$. If $\beta \in O^+\alpha$, then $O^+(\beta) \subset O^+(\alpha)$.*

Proof. Let $\gamma \in O^+(\beta)$. Then there exists a path $\{\beta, x_1, x_2, \ldots, \gamma\}$ from β to γ. Similarly, since $\beta \in O^+(\alpha)$ there exists a path $\{\alpha, \ldots, \beta\}$ from α to β. Simply concatenating these paths constructs a path from α to γ, and thus $\gamma \in O^+(\alpha)$. □

Corollary 13. *For any vertex $\alpha \in V$, the positive orbit of x (that is, the set of vertices y for which there exists a path from x to y) contains some invariant communicating class C.*

Proof. Let $\alpha \in V$. By definition of an N-graph, $o(\alpha) \geq 1$. Thus, there exists a path of length 1 that begins at α. By a similar construction as in the proof of Theorem 10, there thus exist paths of arbitrarily long finite length starting at α, and this positive orbit must contain a communicating class. Let y be a vertex contained within $o(\alpha)$ such that y is contained in a communicating class C. If C is invariant, the proof is done. Thus assume that C is variant. By definition there exists a vertex $z \in C$ such that z is adjacent to a vertex x that is not in C (that is, there exists an edge from z to x). By definition of a communicating class, there exists a path from y to z, and thus a path from y to x, and thus a path from α to x. We further know that the orbit of x contains a communicating class D (and that $D \neq C$, as this would contradict the maximality of both C and D. Once again, if D is invariant, the proof is completed. If D is variant, this process can be repeated once again to locate a communicating class $E \neq D$, $E \neq C$. Since G is a finite graph, this process must terminate. Therefore, by Lemma 12, the positive order of α must contain an invariant communicating class. □

Thus concludes our definitions and theorems on directed graphs. These are the basic principles of N-graphs that will be used throughout the rest of this paper, when forming these dynamical systems.

2.2 Dynamical Systems

We begin by introducing the concept of a dynamical system in general. These systems are often used to model behavior that is seen in biological, economical, medical, and physical systems in addition to many other uses.

Definition 14. *A dynamical system (sometimes abbreviated d.s.) on a metric space X is given by a map $\Phi : \mathbb{T} \times X \to X$ that satisfies $\Phi(0, x) = x$ and $\Phi(t+s, x) = \Phi(t, \Phi(s, x))$ for all $x \in X$ and all $t, s \in \mathbb{T}$. Φ can be expressed by two different but equivalent notations for x, $x' \in X$ and $t \in \mathbb{T}$:*

$$\Phi(t, x) = x' \text{ or } \Phi_t(x) = x'$$

(3).

Definition 15. *A d.s. is 1-sided when $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = \mathbb{R}^+$. A d.s. is 2-sided when $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$.*

A dynamical system can be thought of as a space that is equipped with a "flow" or "movement" that is given by a certain rule. We often require that the map Φ be continuous in the x component. If time takes on real values, we often require that Φ be continuous in the time component as well (this will be trivially true if time takes integer values). Notice that the systems is completely deterministic - everything happens with probability either zero or one. The intrinsic question that governs most study of dynamical systems involves determining limit behavior, or what happens to the system as $t \to \infty$ or $t \to -\infty$.

Lemma 16. *Any 2-sided d.s. with mapping Φ_t has an inverse mapping Φ_{-t} where*

$$\Phi_t \circ \Phi_{-t}(x) = \Phi_{-t} \circ \Phi_t(x) = Id(x) = x$$

As mentioned above, dynamical systems are often studied with the question of limit behavior in mind. The next definitions give a definition of limit structures within a dynamical system.

Definition 17. *The ω-limit set of an element $x \in X$ is*

$$\omega(x) = \{y \in X | (\textit{there exist } t_k \to \infty, k \in \mathbb{N})(\Phi(t_k, x) \to y)\}.$$

The α-limit set of an element $x \in X$ is

$$\alpha(x) = \{y \in X | (\textit{there exist } t_k \to -\infty, k \in \mathbb{N})(\Phi(t_k, x) \to y)\}.$$

Definition 18. *The ω-limit set of a subset $Y \subseteq X$ is given by*

$$\omega(Y) = \{y \in X | (\textit{there exist } t_k \to \infty, y_k \in Y, k \in \mathbb{N})(\Phi(t_k, y_k) \to y)\}.$$

The α-limit set of a subset $Y \subseteq X$ is given by

$$\alpha(Y) = \{y \in X | (\textit{there exist } t_k \to -\infty, y_k \in Y, k \in \mathbb{N})(\Phi(t_k, y_k) \to y)\}.$$

Notice then that the definition of the limit set of a set Y is not simply the union of all the limit sets of points contained in Y. There are other types of sets in dynamical systems that have "nice" properties.

Definition 19. *Given a dynamical system Φ on a metric space X, $A \subset X$ is said to be invariant if for all $x \in A$, $\Phi(t, x) \in A$ for all $t \in \mathbb{R}$.*

Thus, an invariant set can often be considered its own dynamical system set within another dynamical system; it makes sense to consider the invariant set by itself along with the mapping Φ.

However, these are not the only types of sets that have behavior we would like to study. The following definitions are taken from (2), pages 51-56.

Definition 20. *A point $x \in \Omega$ is ω-recurrent if $x \in \omega(x)$, and α-recurrent if $x \in \alpha(x)$. A point $x \in \Omega$ is Poincaré recurrent if x is $\omega-$ and $\alpha-$ recurrent. The Poincaré recurrent set is the set of Poincaré recurrent points.*

Definition 21. *A point $x \in \Omega$ is nonwandering or regionally recurrent if if for each open neighborhood U of x, there exists a natural number $T > 1$ such that*

$$U \cap \Phi_T(U) \neq \emptyset.$$

The nonwandering set is the set of nonwandering points. A wandering point is a point that is not nonwandering.

Proposition 22. *The nonwandering set of a flow is invariant with respect to the flow.*

Proof. A proof can be found in (2), page 57. □

The nonwandering set of any dynamical system displays some convenient qualities, as shown by the next proposition.

Proposition 23. *The nonwandering set of a flow is closed.*

Proof. A proof can be found in (2), pages 57-58. □

It thus follows that the nonwandering set of a dynamical system on a compact space (which will comprise the majority of dynamical systems studied here) are compact as well.
The next type of recurrence that we introduce is called chain recurrence. This type of recurrence relies on a construction known as an (ε, T)-chain, defined below.

Definition 24. *Let Φ^t be a flow on a metric space (Ω, d). Given $\varepsilon > 0$, $T > 0$ and $x, y \in \Omega$, an (ε, T)-chain from x to y with respect to Φ_t and d is a pair of finite sequences $x = x_0, x_1, ..., x_{n-1}, x_n = y$ in Ω and $t_0, ..., t_{n-1}$ in $[T, \infty)$, denoted together by $(x_0, ..., x_n; t_0, ..., t_{n-1})$, such that*

$$d(\Phi_{t_i}(x_i), x_{i+1}) < \varepsilon$$

for $i = 0, 1, 2, ..., n-1$. (2)

Definition 25. *Let Φ_t be a flow on a metric space (Ω, d). The forward chain limit set of $x \in \Omega$ with respect to Φ_t and d is the set $\Psi^+(x) = \bigcap_{\varepsilon, T > 0} \{y \in \Omega |$ there exists an (ε, T)-chain from x to y with respect to $\Phi_t\}$. The backward chain limit set of $x \in \Omega$ with respect to Φ_t and d is the set $\Psi^-(x) = \bigcap_{\varepsilon, T > 0} \{y \in \Omega |$ there exists an (ε, T)-chain from x to y with respect to $\Phi_{-t}\}$* (2)

The use of the intersection over $\varepsilon, T > 0$ requires that *for all* $\varepsilon, T > 0$ there exists an (ε, T)-chain from x to y. Thus, ε may take very small values, while T takes large values.

Definition 26. *Let Φ_t be a flow on a metric space (Ω, d). Two points $x, y \in \Omega$ are chain equivalent with respect to Φ_t and d if $y \in \Psi^+(x)$ and $x \in \Psi^+(y)$.* (2)

Definition 27. *Let Φ_t be a flow on a metric space (Ω, d). A point $x \in \Omega$ is called chain recurrent with respect to Φ_t and d if x is chain equivalent to itself. The set of all chain recurrent points of Φ_t is the chain recurrent set of Φ_t.* (2)

Thus, because chain equivalence and chain recurrence are independent of specific ε and T values, chain equivalence and recurrence really are properties of the flow of the system, rather than tailoring specific ε and T values to create ε, T-chains.

This concludes the introductory chapter on terms, definitions, and well established theorems about graphs, dynamical systems, and recurrence properties. As we go on to create dynamical systems from directed graphs in the next chapter, we will see that recurrence properties in these systems are very much related to the structure within the graphs they are based upon.

CHAPTER 3. SHIFT SPACES - SYMBOLIC DYNAMICS

It is in this chapter that we begin to introduce original work. In order to examine these hybrid dynamical systems, it is critical that we understand the "switching" between the systems as a stand alone object. For this, given a hybrid system $(M \times \Delta, G, \Phi)$, we momentarily forget about the compact space M, and instead examine a space we denote as Ω, which is known as the shift space over G (it is later shown that this space is isometrically isomorphic to the function space Δ, discussed later). It will be shown that the structure of the dynamical system on Ω is very closely tied to the structure of G. This examination also generalizes the concept of symbolic dynamics and chaos that is found with the shift on the bi-infinite product space of the set $\{0, 1\}$, which has been shown to be topologically conjugate to the Smale Horseshoe (see (7), pp 75-80). We will demonstrate that the entire structure of the dynamical system, including a finest Morse Decomposition (defined below) can be determined, and that in many circumstance the system will demonstrate chaotic behavior. We'll also examine some of the recurrence concepts discussed in the previous chapter in the context of Ω, such as chain recurrence, nonwandering sets, and Poincaré sets.

3.1 Introduction to Shift Spaces

The first concept introduced is that of the shift space, the space upon which the dynamics of the symbolic dynamical system takes place.

Definition 28. *Given an N-graph $G = (V, E)$, the bi-infinite product space Υ_G of the set $V = \{1, ..., n\}$ is the set of all bi-infinite sequences $x = (...x_{-1}, x_0, x_1, ...)$ where $x_i \in V$ for all $i \in \mathbb{Z}$.*

Definition 29. *Given an N-graph $G = (V, E)$ with $A \subset V$ and $\alpha \in V$,*

$$\Omega_G = \{ \ (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \mid (x_i, x_{i+1}) \in E \ \}$$

is the shift space of G.

In the case where G is is evident by context, we will simply refer to these sets as Ω and Υ.

The difference between Υ and Ω is that, while Υ contains *all* the bi-infinite sequences made up of vertices of G, Ω restricts these sequences to those "allowed" by the graph; that is, x_i can only follow x_{i-1} if there is an edge leaving x_{i-1} and entering x_i. These sequences in Ω can also be thought of as bi-infinite paths in G. We are guaranteed the existence of these paths because we are considering only N-graphs; since each vertex has positive in- and out-degree, the path will never get "stuck" at any one vertex, either forwards or backwards.

Note that what is happening with the definition of Ω is that we are, in a way, considering dynamics upon the graph G, where a flow is given by movement between the vertices at discrete time intervals; however, in order to define a true dynamical system that satisfies Definition 4.1, we must define a metric space in the form of Ω.

The flow on this dynamical system is determined by the left shift mapping Φ. That is, if $x = (x_i)_{i \in \mathbb{Z}}$, then $(\Phi_1(x))_i = x_{i+1}$. It is important then to note that the indexing on a sequence matters. For example, if entries of a sequence x consisted of all 1s except for one entry being equal to 0, the index of the location of the 0 will determine the sequence; otherwise, there are infinitely many sequences that could fit the above description, without specifying the location of the 0.

Definition 30. *Given a set $A \subset V$, $\Omega_A = \{ \ x \in \Omega \mid x_i \in A$ for all $i \in \mathbb{Z} \ \}$ is called the lift of A.*

We will be particularly concerned with lifts of communicating classes. Thus, the lift of a communicating class C can be thought of as "all possible sequences whose entries are contained in C."

Definition 31. *Given a vertex $\alpha \in V$, $\Omega_\alpha = \{x \in \Omega | x_0 = \alpha\}$.*

Recognizing that this notation can be a bit confusing, it is important to differentiate between Ω_α and $\Omega_{\{\alpha\}}$. The former is defined above in Definition 31. The latter is, by Definition 30, equal to $\{x \in \Omega | x_i = \alpha \text{ for all } i \in \mathbb{Z}\}$. Therefore, this set either consists of a single point, if α has a self loop, or is empty, if α does not have a self loop.

Lemma 32. *All non-empty lifts are Φ-invariant.*

Proof. Let $A \subset V$ be non-empty and Ω_A be its non-empty lift. Suppose there exists some $x \in \Omega_A$ such that $\Phi(x) = x' \notin \Omega_A$. Since $x' \notin \Omega_A$ then for some $i \in \mathbb{Z}$, $\pi_i(x') \notin A$ or for some $x_i, x_{i+1} \in A$, $(x'_i, x'_{i+1}) \notin E$. But if $\Phi(x) = x'$ then $\pi_{i-1}(x) = \pi_i x' \notin A$ so $x \notin \Omega_A$, and if $(x'_i, x'_{i+1}) \notin E$, then $(x_{i-1}, x_i) \notin E$ implying $x \notin \Omega$, contradicting that $x \in \Omega_A$. $\square$

Thus, note that because the lifts of communicating classes in an N-graph are non-empty, they are invariant. In addition, since by Theorem 10, any N-graph G contains a communicating class, for all N-graphs G, Ω_G must contain a lift of a communicating class (in fact, there are some cases where $\Omega = \Omega_C$ for a communicating class C, namely when G itself is an entire communicating class, and thus $G = C$).

Lemma 33. *For an N-graph $G = (V, E)$ and $\alpha \in V$, $O^+(\alpha) = \bigcup_{i \in \mathbb{N}} \{\pi_i(x) | x \in \Omega_\alpha\}$*

Proof. ($\subseteq$) Let $\beta \in O^+(\alpha)$. Then there exists $\gamma \in P$ such that $\gamma_0 = \alpha$ and $\gamma_i = \beta$ for some $i \in \mathbb{N}$. Since Ω_α is made up of all points in Ω which are admissible sequences that have an $x_0 = \alpha$, there must be some $x \in \Omega_\alpha$ which has $x_i = \beta$ for some $i \in \mathbb{N}$ because γ is an admissible path in G. Thus $\beta \in \bigcup_{j \in \mathbb{N}} \{\pi_j(x) | x \in \Omega_\alpha\}$.

($\supseteq$) Now let $\beta \in \bigcup_{i \in \mathbb{N}} \{\pi_j(x) | x \in \Omega_\alpha\}$. Then there exists some $x \in \Omega$ which is an admissible sequence and has $x_0 = \alpha$ and $x_i = \beta$ for some $i > 0$. Thus if there is an admissible forward sequence from α to β, by definition it must be true that there is an admissible path in G from α to β so $\beta \in O^+(\alpha)$. $\square$

Lemma 34. *Given $A, B \subset V$ communicating classes of G: $\Omega_A \cap \Omega_B = \emptyset$.*

Proof. Since A and B are communicating classes in G, by maximality they are disjoint. Thus by the definition of lifts, $\Omega_A \cap \Omega_B = \emptyset$. □

We now move on to discussing the topology for Ω, including a metric and an overview of the properties of some of its subsets.

3.2 Characterizing the Shift Space

In order to discuss limit sets and recurrence properties of Ω, Φ, it is important to have a way of discussing the topology of Ω. In this section a metric is first defined on the space Ω. Once this metric has been shown to be satisfactory the discussion will move more generally to the topology it induces on Ω. Important topics covered will be those of completeness, closed subsets, and compactness, all of which will be immensely useful in the last two subsections.

Definition 35.

$$f(x_i, y_i) = \begin{cases} 0 & \textit{when } x_i = y_i \\ 1 & \textit{when } x_i \neq y_i \end{cases}$$

Lemma 36. *The function $d(x,y) = \displaystyle\sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}}$, is a metric on Υ.*

Proof.

1. (**Non-negativity**) For any $x, y \in \Upsilon$ and all $i \in \mathbb{Z}$, $\frac{f(x_i,y_i)}{4^{|i|}}$ is a quotient of two nonnegative numbers, so it is nonnegative. Nonnegative reals are closed under addition, so $d(x,y) \geq 0$.

2. (**Identity of Indiscernables**) Suppose $x \neq y$. Then there is some i such that $x_i \neq y_i$, so $f(x_i, y_i) = 1$. Thus, $d(x,y) \geq \frac{1}{4^{|i|}} > 0$. If $x = y$, then $x_i = y_i$ for all i, and $f(x_i, y_i) = 0$ for all i. So then $d(x,y) = \displaystyle\sum_{i=-\infty}^{\infty} 0 = 0$. So $d(x,y)$ is zero if and only if $x = y$.

3. (**Symmetry**) Clearly, $f(a,b) = f(b,a)$ for all $a, b \in V$, so $d(x,y) = \displaystyle\sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} = \displaystyle\sum_{i=-\infty}^{\infty} \frac{f(y_i, x_i)}{4^{|i|}} = d(y,x)$.

4. (**Triangle Inequality**) First, we need to show $f(k,j)+f(j,l)\geq f(k,l)$. If $k=l$, the case is trivial. If $k\neq l$, then $f(k,l)=1$. Consider two cases- $j=k$ or $j\neq k$. If $j=k$, then $j\neq l$, and $f(j,l)=1$. Then $f(k,j)+f(j,l)=1\geq f(k,l)$. If $j\neq k$, then $f(k,j)=1$, so $f(k,j)+f(j,l)\geq 1=f(k,l)$. Thus, $d(x,y)+d(y,z)=\displaystyle\sum_{i=-\infty}^{\infty}\frac{(f(x_i,y_i)+f(y_i,z_i))}{4^{|i|}}\geq \displaystyle\sum_{i=-\infty}^{\infty}\frac{f(x_i,y_i)}{4^{|i|}}=d(x,z)$.

□

The topology induced by using 4 in the denominator is slightly different from those induced by 2 or 3. In the case of 4, two points in Ω that agree in the i^{th} position, but are different before and beyond, will have a distance less than two points that disagree in the i^{th} position and agree before and beyond, which is not necessarily the case with 2 or 3. All integers greater than 4 will induce an equivalent metric to that induced by 4, so the choice to use 4 was made for simplicity. We now examine some topological properties of Ω and Ω_C.

Lemma 37. *All Cauchy sequences in Ω converge to some x in Υ.*

Proof. Let $\{x^n\}_{n=1}^{\infty}$ be a Cauchy sequence in Ω. Given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $d(x^m,x^n)<\varepsilon$ for all $n,m>N$. For all $i\in\mathbb{Z}$, there exists $\varepsilon'>0$ such that $\frac{1}{4^i}>\varepsilon'$. Then there exists $N'\in\mathbb{Z}$ such that $x_i^m=x_i^n$ for some $i\in\mathbb{Z}$ for all $n,m>N'$. Therefore, $\lim_{n\to\infty}x_i^n$ exists for all $i\in\mathbb{Z}$, and is in V. Let $y=(...,\lim_{n\to\infty}x_{-1}^n,\lim_{n\to\infty}x_0^n,\lim_{n\to\infty}x_1^n,...)\in\Upsilon$. Then for all $\varepsilon>0$, there exists $j\in\mathbb{N}$ such that $\sum_{i=j}^{\infty}\frac{2}{4^i}<\varepsilon$. There exists $N_{-j},N_{-j+1},...,N_j$ such that $y_i=x_i^n$ for all $n>N_i$, $i\in[-j,j]$. Let $N=\max\{N_{-j},...N_j\}$. So for $n>N$, $x_i^n=y_i$, and so

$$d(x^n,x)<\sum_{i=j}^{\infty}\frac{2}{4^i}<\varepsilon.$$

Thus, $\{x^n\}_{n=1}^{\infty}$ converges to y. □

Lemma 38. *The shift operator Φ_t is continuous for all t (that is, $\Phi(t,x)$ is continuous in the x component.*

Proof. If we show Φ_1 is continuous, then Φ_t is continuous by the continuity of continuous compositions. Let $\varepsilon > 0$ be given. We claim that there exists $\delta > 0$ such that if for two sequences $x, y \in \Omega$, if $d(x,y) < \delta$, $d(\Phi_1(x), \Phi_1(y)) < \epsilon$. Let $\delta = \varepsilon/4$. Pick any $x, y \in \Omega$ such that

$$d(x,y) = \sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} < \delta$$

$$d(\Phi_1(x), \Phi_1(y)) = \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i|}}$$

So we can then form:

$$(1/4)d(\Phi_1(x), \Phi_1(y)) = \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i|+1}} \leq \sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i+1|}}$$

We then know

$$\sum_{i=-\infty}^{\infty} \frac{f(x_{i+1}, y_{i+1})}{4^{|i+1|}} = \sum_{i=-\infty}^{\infty} \frac{f(x_i, y_i)}{4^{|i|}} = \delta$$

Which then yields:

$$d(\Phi_1(x), \Phi_1(y)) \leq 4\delta = \varepsilon$$

. □

We now introduce the important concept of a cylinder set. This relatively simple idea is useful further on to demonstrate properties of Ω.

Definition 39. *Given $x \in \Omega$, $n \in \mathbb{N}$, the cylinder set of length $2N+1$ around x is given by* $\{y \in \Omega | y_i = x_i, i \in \{-N, -N+1, \ldots, -10, 1, \ldots N-1, N\}\}$

Thus, given a point $x \in \Omega$ and a natural number N, the cylinder set of length $2N+1$ around x is the set of all points that share those same middle $2N+1$ entries; thus, any point in the cylinder set can be used as the center of the set. Also notice that, if $M \geq N$, then the cylinder set of length $2M+1$ around x is contained in the cylinder set of length $2N+1$.

Lemma 40. *For all $x \in \Omega$, $\varepsilon > 0$, $B(x, \varepsilon)$ contains a cylinder set of 2N+1 fixed elements about the origin for some N.*

Proof. Let $x \in \Omega$, $\varepsilon > 0$. By definition, $B(x, \varepsilon)$ contains all points y such that $d(x, y) < \varepsilon$. Because $\displaystyle\sum_{i=-\infty}^{\infty} \frac{1}{4^{|i|}}$ is a convergent series, there exists an N such that

$$\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} + \sum_{i=-\infty}^{-N} \frac{1}{4^{|i|}} = 2\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

Let D^N represent the cylinder set of length $2N+1$ around x. Let $y \in D^N$ Then,

$$d(x, y) \leq 2\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

Thus, $y \in B(x, \varepsilon)$, and thus $D^N \subset B(x, \varepsilon.)$ $\square$

This last lemma insures that in any ε-ball of point $p \in \Omega$ it is possible to find other points in the ball which have the same entries as x from entries x_{-N} to x_N for some N. This will be most useful when discussing convergent sequences and especially when it is necessary to construct points with certain properties in ε-balls in Ω. All that will need to be done is form a ε-ball and show that it is possible find a point within that ball which is equal to all other points in the ball for at some fixed $-N$ to N values but contains a constructed sequence before or beyond those entries.

Lemma 41. *Ω is closed in Υ.*

Proof. Assume we have a convergent sequence $\{x^n\}_{n=1}^{\infty} \in \Omega$ such that $\lim_{n\to\infty} x_n = y$. Assume that $y \notin \Omega$. Then there exist $i \in \mathbb{Z}$ such that $(y_i, y_{i+1}) \notin E$. Note that there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4^{|i|}}$, $\varepsilon < \frac{1}{4^{|i+1|}}$. Thus, there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_i^n = y_i$, and $x_{i+1}^n = y_{i+1}$. Therefore, for all $n > N$, $(x_i^n, x_{i+1}^n) \notin E$, implying for all $n > N$, $x^n \notin \Omega$, contradicting that the sequence $\{x^n\}_{n=1}^{\infty}$ is a sequence in Ω. $\square$

Corollary 42. *Ω is complete.*

Proof. Ω is closed, and all Cauchy sequences in Ω converge, so all Cauchy sequences in Ω converge to a point in Ω. $\square$

Lemma 43. *Ω is totally bounded.*

Proof. Given $\varepsilon > 0$, we need to show that there exist a finite collection of ε-balls covering Ω. Take N such that

$$\sum_{i=-\infty}^{-N} \frac{1}{4^{|i|}} + \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

This ensures that any cylinder set of order N will be contained in a ball of radius ε. If the graph generating Ω has M vertices, then there exist at most M^{2N+1} possible cylinder sets of order N in ε. Take a collection of M^{2N+1} ε-balls, each of which covers a cylinder set of order N. Clearly, this collection of ε-balls is finite. And, since any element of Ω is contained in some cylinder set of order N, the collection of ε-balls covers Ω. Hence, Ω is totally bounded. □

These results lead naturally to the compactness of Ω.

Theorem 44. *Ω is compact.*

Proof. By Corollary 42 and Lemma 43, Ω is complete and totally bounded, and thus is compact. □

Now that it is known that Ω is a compact set, when we consider the flow Φ on Ω, we know that α, ω-limit sets are nonempty, and we can apply other well known theorems about dynamical systems on compact metric spaces.

Lemma 45. *For any $C \subset V$ a communicating class in G: Ω_C is closed.*

Proof. Assume we have a convergent sequence of $\{x^n\}_{n=1}^{\infty} \in \Omega_C$ such that $\lim_{n\to\infty} x^n = x$. In order to prove that Ω_C is closed, we have to show $x \in \Omega_C$. Because the sequence is convergent to x, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \varepsilon$. Assume $x \notin \Omega_C$. Therefore, there exists an entry of x, x_i, such that $x_i \notin C$. Hence $d(x^n, x) \geq \frac{1}{4^{|i|}}$, contradicting convergence. Thus, $x \in \Omega_C$ and the set is closed. □

Corollary 46. *For any $C \subset V$ a communicating class in G: since Ω_C is a closed subset of the compact space Ω, it follows that Ω_C is compact.*

This leads us to further realize that since Ω_C is compact and invariant, any α, ω-limit sets of points within Ω_C must be contained within Ω_C.

Finally, we will prove the following lemma on periodic points in Ω. Its significance will become relevant later when the topic of chaos is examined more closely.

Lemma 47. *Given G and associated Ω: points of Ω_p are dense in the lifts of communicating classes.*

Proof. Pick $x \in \Omega_C$ and form an ε-ball $B(x, \varepsilon)$ about x for $\varepsilon > 0$. By Lemma 40, $B(x, \varepsilon)$ contains a cylinder set of 2N+1 fixed elements about the origin for some $N \in \mathbb{N}$. Therefore, a periodic sequence of period 2N+1+k exists such that the repeated portion is the finite subsequence $(x_{-N}, ..., x_0, ..., x_N, ..., x_{N+k})$ where $x_{N+k} \in V$ and $(x_{N+k}, x_{-N}) \in E$. A periodic sequence of this form can always be found because x_{-N} and x_N are in a communicating class together, thus there does exist a path from x_N to x_{-N}. Hence, because for all $\varepsilon > 0$, $B(x, \varepsilon)$ contains an element of Ω_p, points of Ω_p are dense in Ω_C. □

Now that a metric and its induced topology on Ω have been defined and to some degree described it will be useful to take a more thorough look at how Φ affects points in Ω.

Proposition 48. *For all $x \in \Omega$, $\omega(x)$ and $\alpha(x)$ are closed sets in Ω.*

Proof. A proof can be seen in (2). □

Definition 49. *For $\Omega_A, \Omega_B \subset \Omega$, a cycle between Ω_A and Ω_B exists if there exist some $p, q \in \Omega$ such that $\omega(p) \subset \Omega_A, \omega(q) \subset \Omega_B$ and $\alpha(p) \subset \Omega_B, \alpha(q) \subset \Omega_A$.*

This definition can be further extended to cycles between three or four lifts, or between any finite number of lifts.

Cycles become very important when Morse Decompositions for Ω are discussed in Section 3.5. Before discussing where these cycles can and cannot exist in Ω, however, it is important to first understand what it means for a point to converge forwards and backwards to two different lifts:

Lemma 50. *Given $A, B \subset V$ and their non-empty lifts $\Omega_A, \Omega_B \subset \Omega$, there exists some point $p \in \Omega$: $\alpha(p) \subset \Omega_A$ such that $\omega(p) \subset \Omega_B$ if and only if there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in A$ and $\gamma_F \in B$.*

Proof. ($\Rightarrow$). Assume $\alpha(p) \subset \Omega_A$ and $\omega(p) \subset \Omega_B$. Then there exist $i, j \in \mathbb{Z}$, with $i < j$ such that for all $t < i$, $\pi_t(p) \in A$ and for all $t > j$, $\pi_t(p) \in B$. Since $p \in \Omega$, p is an admissible bi-infinite path of G, so the subsequence $(p_{i-1}, ..., p_{j+1})$ of p is also an admissible path of G. Thus, if we let $\gamma = (p_{i-1}, ..., p_{j+1})$, then $\gamma_1 \in A$ and $\gamma_F \in B$.

($\Leftarrow$). Assume there exists $\gamma \in \mathcal{P}$ with $\gamma_1 \in A$ and $\gamma_F \in B$ and let γ have length $k+1$. Since Ω_A and Ω_B are non-empty, and since all non-empty lifts are Φ-invariant we can find some $x \in \Omega_A$ and $y \in \Omega_B$ such that $\pi_0(x) = \gamma_1 \in A$ and $\pi_0(y) = \gamma_F \in B$. Then it is possible to construct $p = (..., \gamma_1, ..., \gamma_F, ...)$ with $\pi_i(p) = \pi_i(x)$ for all $i \leq 0$, $\pi_{k+i}(p) = \pi_i(y)$ for all $i \geq 0$, and $\pi_{i-1}(p) = \gamma_i$ for all $1 \leq i \leq k+1$. Then clearly $\alpha(p) \subset A$, $\omega(p) \subset B$, and $p \in \Omega$ since p is made up entirely of admissible paths in $\mathcal{P}$. $\square$

Notice now that the behavior of trajectories in Ω is very closely related to the general flow of paths through the directed graph G. In fact, forward motion is very comparable to forward paths on G, and the same for backwards motion. Thus, we use the next section to discuss the relationship between behavior in the directed graph and behavior, recalling some characteristics of communicating classes that exist within the context of the graph.

Lemma 51. *Given an N-graph with at least two communicating classes, there are no cycles between the lifts of the communicating classes.*

Proof. Consider two distinct communicating classes $A, B \subset V$ and construct their lifts as Ω_A and Ω_B. Suppose their exists $p, q \in \Omega$ such that $\omega(p) \subset \Omega_A, \omega(q) \subset \Omega_B$ and $\alpha(p) \subset \Omega_B, \alpha(q) \subset \Omega_A$. Then this would constitute a cycle between Ω_A and Ω_B. Since the point p is an admissible sequence representing an an admissible path in G, we know there must exist a path from vertices in B to vertices in A because entries in the left tail of p must be contained in A and entries in the right tail of p must be contained in B. In addition since the point q is also an admissible sequence representing an admissible path in G, we know there must also exist a path from vertices in A to vertices in B. The existence of these paths makes the set $A \cup B \subseteq V$ a communicating class in V, which contradicts the maximality of A and B. Thus there can be no cycle between Ω_A and Ω_B. $\square$

One can see that is lemma can easily be extended to cycles between any finite number of lifts of communicating classes, as the existence of any of these would contradict the maximality of communicating classes. In fact, as is seen later on, the arrangement of communicating classes within a graph lends itself nicely to a partial ordering.

From this lemma we can begin to see what two points in Ω cannot do. We now ask the opposite question for a single point in Ω. In this next lemma the limit sets of all points in Ω are shown to been completely contained by the lifts in Ω of communicating classes in the initial graph.

Lemma 52. *For all $x \in \Omega$, if an N-graph G has communicating classes $C_1, ..., C_k$,*

$$\alpha(x), \omega(x) \subset \bigcup_{i=1}^{k} \Omega_{C_i}.$$

In particular, each $\alpha(x)$ and $\omega(x)$ are wholly contained in one specific Ω_{C_i}.

Proof. Let $a \in \pi(\omega(x))$. Then there exists a $y \in \omega(x)$ and an $N \in \mathbb{Z}$ such that $y_N = a$. Since $y \in \omega(x)$, there exists a sequence of natural numbers $n_k \to \infty$ such that $\Phi_{n_k}(x) \to y$. Recall that

$$\Phi_{n_k}(x) = \Phi_{n_k}(x_i, i \in \mathbb{Z}) = (x_{i+n_k}, i \in \mathbb{Z}).$$

Note that there exists $\varepsilon > 0$ such that $\varepsilon < \frac{1}{4^{|N|}}$. Thus, there exists K such that $\pi_N(\Phi_{n_k}(x)) = a$, for all $k > K$, that is, $\pi_N(x_{i+n_k}, i \in \mathbb{Z}) = a$ for all $k > K$. So for all i such that $i = N + n_k$ for $k > K$, $x_i = a$. Since $n_k \to \infty$, for all $n \in \mathbb{Z}$, there exists n_k such that $N + n_k > n$.

Thus, if a vertex a is in $\pi(\omega(x))$, there exists a path $(x_{N+n_k}, x_{N+n_k+1}, \ldots, x_{N+n_{k+1}})$ in G from a to itself. Thus, there is a loop through a, and so by Theorem 21, a is in a communicating class.

Now let $a, b \in \pi(\omega(x)$. Then there exists n_1 such that $\pi_{n_i}(x) = a$ and $m > n_1$ such that $\pi_m(x) = b$. There also exists $n_2 > m$ such that $\pi_{n_2}(x) = a$. Thus, there is a path $(x_{n_1}, x_{n_1+1}, \ldots, x_m)$ a path from a to b in G and a path $(x_m, x_{m+1}, \ldots, x_{n_2})$ a path from b to a in G. Thus, a and b must be elements of the same communicating class. $\square$

Theorem 53. *Let $X = \{x \in \Omega \mid \omega(x) \subset \Omega_C$ where $C \subset V$ is some invariant communicating class of $G\}$. Then X is open and dense in Ω.*

Proof. First, we will show that X is dense in Ω. Consider any $y \in \Omega$, and let $\varepsilon > 0$. Then there exists an N such that $\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$. There is an invariant communicating class in the positive orbit of every vertex in an N-graph by Corollary 13, and so there exists a path $\gamma = \gamma_1, ..., \gamma_k$ with $\gamma_1 = y_N$ and $\gamma_k = z$ where z is an element of some invariant communicating class. Thus, there is some point x such that $x = (..., y_{N-1}, y_N, \gamma_2, ..., \gamma_k, \gamma_1', ...)$ where γ' is some infinite path in the invariant communicating class containing z. Then $d(y, x) \leq \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon$, so $x \in B(y, \varepsilon)$. Also, $x \in X$, and thus because y and ε were arbitrary, X is dense in Ω.

Now consider any $x \in X$. There must exist some N such that $x_N \in C$ an invariant communicating class of G. Let $\varepsilon = \frac{1}{4^{N+1}}$. Then for any $y \in B(x, \varepsilon)$, $y_i = x_i$ for all $i = -N, -N+1, ..., N$, so y_N is in C. Thus, $y_M \in C$ for all $M > N$, and so $\omega(y) \subset \Omega_C$. So $y \in X$. Thus, $B_\varepsilon(x) \subset X$, and so X is open in Ω. $\square$

The last definition and the lemma that follows it have importance which will become clearer in the next section. For this section, however, it should suffice to see that, using limit sets, we can characterize the action of the shift, Φ, on the lifts of communicating classes as being topologically transitive.

Definition 54. *A flow on a metric space X is topologically transitive if there exists some $x \in X$ such that $\omega(x) = X$.* (3)

Lemma 55. *Φ is topologically transitive on the lifts of communicating classes; that is, for each communicating class C_i there exists a point x^* such that $\omega(x^*) = \Omega_{C_i}$.*

Proof. Consider a communicating class C_i, and consider all loops in C_i of length n, denoted by $A_n^1, \dots,\ A_n^{k_n}$. Let $x \in \Omega$ be constructed by the concatenation of all loops, connected by paths linking the last vertex of each loop to the first of the next (which we know to exist, because C_i is a communicating class), starting with those of length 1. That is,

$$x = (\cdots \circ A_n^{k_n} \circ \cdots \circ A_n^1 \circ \cdots \circ A_1^2 \circ p_i \circ A_1^1 \circ p_j \circ A_1^2 \circ \cdots \circ A_n^1 \circ \cdots \circ A_n^{k_n} \circ \dots).$$

We claim that $\omega(x)$ contains every periodic sequence in Ω_{C_i}. It suffices to show that every periodic sequence is a subsequence of x. Let p be a periodic sequence in Ω_{C_i} with period m. Thus, there exists a loop A in C_i of length m such that $p = (\cdots \circ A \circ A \circ A \circ A \circ \dots)$. Notice also that the concatenation of A twice is a loop of length $2m$ in C_i, and similarly for all $l \in \mathbb{N}$. Thus, this loop of length m appears infinitely throughout x in both directions, implying that p is a bi-infinite subsequence of x, and thus $p \in \omega(x)$.

Because periodic points are dense in Ω_{C_i}, and $\omega(x)$ is closed, $\omega(x) \subseteq \Omega_{C_i}$, but since $\omega(x)$ is clearly contained in Ω_{C_i}, $\omega(x) = \Omega_{C_i}$. $\square$

Corollary 56. *This proof also insures that $\alpha(x^*) = \Omega_C$.*

Because this point occurs in the lifts of communicating classes we can use the fact that paths exist between all vertices in any sequence and a similar technique to that used in Theorem 53 to show an extension of the same concept, namely that points for which $\omega(x) = \Omega_C$ are dense in Ω_c.

Lemma 57. *Given an N-graph, G, with communicating class C, lift Ω_C and constructed point $x^* \in \Omega_C$ for which $\omega(x^*) = \Omega_C$, let $Y = \{\ x \in \Omega_C \mid \omega(x) = \Omega_C\ \}$. Then Y is dense in Ω_C.*

Proof. Consider $p \in \Omega_C$, $\varepsilon > 0$, and form $B(p, \varepsilon)$. Then we know there exists some $N + 1 \in \mathbb{N}$ such that

$$2 \sum_{N+1}^{\infty} \frac{1}{4^i} < \varepsilon,$$

so all points $q \in \Omega_C$ for which

$$\sum_{-N}^{N} \frac{f(p_i, q_i)}{4^{|i|}} = 0$$

are in $B(p, \varepsilon)$. Then, since $\pi_N(p) \in C$ and since for all $i \in \mathbb{Z}$, $\pi_i(x^*) \in C$ from the constructed x^*, we know there exists $\gamma \in \mathcal{P}$ of some length k for which $\gamma_1 = \pi_N(q)$ and $\gamma_F = \pi_0(x^*)$. Then it is possible to form $q^* \in B(p, \varepsilon)$ such that $q_i^* = p_i$ for all $-N \leq i \leq N$, $q_i^* = x_i^*$ for all $N + k \leq i$, and $q_i^* = \gamma_i$ for all $N \leq i \leq k$. Then, with this construction $\omega(q^*) = \omega(x^*) = \Omega_C$ and $q^* \in B(p, \varepsilon)$. □

Corollary 58. *Not only are points for which $\omega(x) = \Omega_C$ dense in Ω_C, but points for which $\alpha(x) = \Omega_C$ and for which both $\alpha(x), \omega(x) = \Omega_C$ are also dense in Ω_C.*

Finally, now that we have an understanding of the behavior of the flow on the lifts of communicating classes, we are able to sum up all of these properties with one term.

Definition 59. *A flow on a metric space X has sensitive dependence on initial conditions if there is $\delta > 0$ such that for every $x \in X$ and every neighborhood B of x there are $y \in B$ and $t > 0$ such that $d(\Phi_t(y), \Phi_t(x)) > \delta$.* (3)

Here we introduce the concept of chaos. Chaos is often, in pop culture, referred to as "The Butterfly Effect". This term is coined from a remark of Edward Norton Lorenz, a pioneer in chaos theory, who once remarked that the flap of a butterfly's wings in Borneo could eventually lead to a hurricane in Florida. That is, having only an approximate understanding of the present, to within any accuracy you like, will eventually not be enough to understand future behavior.

Proving the existence of chaotic behavior is actually made simpler with the below proposition.

Proposition 60. *Consider a flow on a space X that is not a single periodic orbit. If the space is topologically transitive and has a dense subset of periodic points, then it has sensitive dependence on initial conditions, and the behavior is called "chaotic".*

Proof. A proof of this proposition is given in (3). □

Thus we can see that the work on lifts of communicating classes up to this point has lead to demonstrating chaotic behavior on the lifts of communicating classes. This is summed up in the theorem below.

Theorem 61. *For an N-graph, G, and associated shift space Ω: The flow, Φ, has sensitive dependence on initial conditions on the lifts of communicating classes of G that are not comprised of a periodic orbit.*

Proof. From Lemma 55, Φ is topologically transitive on the lifts of communicating classes in G, and from Lemma 47 periodic points are dense in the lifts of communicating classes. Thus, by Proposition 60 the lifts of communicating classes in G have sensitive dependence on initial conditions. □

3.3 Communicating Classes: Poincaré and Chain Recurrence, Nonwandering Sets, Morse Decompositions, and Chaos

In the previous sections we discussed the flow of individual points and characteristics of certain types of subsets of our space Ω. In a sense, this all serves the purpose of attaining a global understanding of the nature of our dynamical system and the entirety of its flow. However, doing so requires more than the previously discussed ω and α limit sets. Thus, we revisit the concepts of recurrence introduced in Chapter 1.

The chain limit sets and chain recurrent points are two more steps towards understanding the flow of the entire system. An educated guess can be made that if all ω and α limit sets are in lifts of communicating classes then these lifts must have some significance. In the next theorem we show how all chain recurrent points are contained in those lifts and define more of their properties that will be of importance for the Morse Decomposition.

Theorem 62. *$x \in \Omega$ is a chain recurrent point if and only if $x \in \Omega_C$ for some $C \subset V$ a communicating class of G.*

Proof. ($\Rightarrow$). Let $x \notin \Omega_C$ for any C, a communicating class of G. Then there exists some combination of $i, j, k \in \mathbb{Z}$ with either $\pi_i(x)$ a variant vertex or $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$, communicating classes of G, where $C_1 \neq C_2$ and $j < k$. We must show that for some ε, T combination there does not exists an (ε, T)-chain from x to itself.

For the case where there is some $\pi_i(x) \notin C$ for any communicating class, fix $T > 0$ such that $T > i$ and let $\Phi_T(x) = x'$. Then for all $m \geq i$, $\pi_m(x') \in O^+(\pi_i(x))$ and $x'_{i-T} = x_i$. Then for all $\varepsilon < \frac{1}{4^{|m|}}$, $q \in B(x', \varepsilon)$ has $\pi_m(q) \in O^+(\pi_i(x))$ for all $m \geq i$ and $q_{i-T} = x_i$. Then all points, $q' \in \Psi^+(q)$, reachable by sequences of T's and ε-jumps from q, will have $\pi_i(q') \in O^+(\pi_i(x))$. This will not allow a point $p \in \Omega$ with $p_i = x_i$ because $\pi_i(x)$ is variant, i.e. there is no loop between vertices of $O^+(\pi_i(x))$ and $\pi_i(x)$. Thus $x \notin \Psi^+(x)$.

For the case where there is some $\pi_j(x) \in C_1$ and $\pi_k(x) \in C_2$ where $C_1 \neq C_2$ pick the initial T such that $k - T = j$ and form $\Phi_T(x) = x'$. Then for all $m \geq j$, $\pi_m(x') \in O^+(\pi_k(x))$, $x'_{j-T} = x_j$, and $x'_{k-T} = x'_j = x_k$. So for all $\varepsilon < \frac{1}{4^{|j|}}$, $q \in B(x', \varepsilon)$, has $q_j = x_k$ for all $m \geq j$. Then, similar to before, all points $q' \in \Psi^+(q)$, reachable by sequences of T's and ε-jumps from q will have $\pi_j(q') \in O^+(\pi_k(x)$. So, again, no point $p \in \Omega$ with $p_j = x_j$ will be reached because by the definition of communicating class there will be no loops between elements of C_1 and $O^+(C_2)$. Thus, as before, $x \notin \Psi^+(x)$.

($\Leftarrow$). Let $a = (...a_{-1}, a_0, a_1, ...)$ and $c = (...c_{-1}, c_0, c_1, ...)$ with $a, c \in \Omega_C$. Given $\varepsilon, T > 0$ there must exists some N such that

$$\sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \frac{\varepsilon}{10}.$$

Let $M = \max(N, T)$. Then let $a' = \Phi_M(a)$. By the definition of communicating class, there is a path $\gamma \in P$ with $\gamma_1 = a'_{M+1}$ and $\gamma_F = c_{-M-1}$. Let k be the length of γ. Then we can construct $e \in C$ such that $e = (...a'_{-1}, a'_0, ...a'_M, \gamma_1, ..., \gamma_F, c_{-M}, c_{-M+1}, ...)$ with a'_0 at the origin to be an admissible sequence. The distance then, between a and e is

$$d(a', e) \leq \sum_{i=M+1}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

So we can construct an ε-T chain from a to e. Similarly we can construct $e' \in C$ such that $e' = \Phi_{2M+k+1}(e) = (...a'_{-1}, a'_0, ...a'_M, \gamma_1..., \gamma_F, c_{-M}, c_{-M+1}, ...c_0, ...)$ with c_0 at the origin to be an admissible sequence. This gives us

$$d(c, e') \leq \sum_{i=-\infty}^{-M-1} \frac{1}{4^{|i|}} < \varepsilon.$$

So we can make an ε-T chain from e to c, and thus we can make an $\varepsilon - T$ chain from a to c. Since a and c are arbitrary, it follows that all points in the lift of a communicating class are chain recurrent. $\square$

Thus we understand all of the chain recurrent points of (Ω, Φ). There are other nice recurrence properties that the lifts of communicating classes exhibit as well, such as isolation.

Definition 63. *An invariant subset $R \subset \Omega$ is said to be isolated if there exists a neighborhood N of R such that for all $x \in \Omega$, if $\omega(x) \subset N$ and $\alpha(x) \subset N$, then $x \in R$.* (3)

In order to prove the isolation of lifts of communicating classes, we first introduce a partial order on the lifts of communicating classes, as was hinted at previously.

Definition 64. *Let $\mho_S = \{\Omega_C | C$ is a communicating class of $A\}$. We define $\prec$ on $\mho_S$ as: for $\Omega_1, \Omega_2 \in \Omega_S$, $\Omega_1 \prec \Omega_2$ when there exists $p \notin \Omega_1, \Omega_2$ such that $\omega(p) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1$.*

Lemma 65. *$\prec$ is an order on $\mho_S$.*

Proof.

- $\prec$ is asymmetric: suppose there were $\Omega_1, \Omega_2 \in \mho_S$ such that $\Omega_1 \prec \Omega_2$ and $\Omega_2 \prec \Omega_1$. Then there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_1, \omega(q) \subset \Omega_2$ and $\alpha(p) \subset \Omega_1, \alpha(q) \subset \Omega_2$. But this constitutes a cycle between Ω_1 and Ω_2, which is a contradiction.

- $\prec$ is transitive: assume we have $\Omega_A, \Omega_B, \Omega_C \in \mho_S$ with $\Omega_A \prec \Omega_B$ and $\Omega_B \prec \Omega_C$. So there exists points $p, q \in \Omega$ such that $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ and $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$. Since $\omega(p) \subset \Omega_B, \alpha(p) \subset \Omega_A$ there exists some $N_1, N_2 \in \mathbb{Z}$ such that $\pi_{N_2+k}(p) \in B, \pi_{N_2-k}(p) \notin B$ and $\pi_{N_1-k}(p) \in A, \pi_{N_1+k}(p) \notin A$ for all $k \in \mathbb{N} \cup \{0\}$. Similarly since $\omega(q) \subset \Omega_C, \alpha(q) \subset \Omega_B$ there exists some $N_3, N_4 \in \mathbb{Z}$ such that $\pi_{N_4+k}(p) \in C, \pi_{N_4-k}(p) \notin C$ and $\pi_{N_3-k}(p) \in B, \pi_{N_3+k}(p) \notin B$ for all $k \in \mathbb{N} \cup \{0\}$. Now construct the point p^* to have $\pi_{-k-i}(p^*) = \pi_{N_2-i}$ for all $i \in \mathbb{Z}^-$, and $\pi_{k+i}(p^*) = \pi_{N_3+i}$ for all $i \in \mathbb{Z}^+$. We leave the values $\pi_i(p^*)$ for $-k \leq i \leq k$ to be filled in by the minimal path between $\pi_{N_2}(p) \in B$ and $\pi_{N_3}(q) \in B$ which we know exists because they are in the same communicating class. So from our construction we can see that $\alpha(p^*) \subset \Omega_A$ and $\omega(p^*) \subset \Omega_C$. Thus $\Omega_A \prec \Omega_C$.

□

Thus, going back to the underlying graph G, once a communicating class has been left, it can not be revisited. This is what guarantees that there will be no cycles between lifts of communicating classes.

We are now ready to prove the isolation of lifts of communicating classes.

Lemma 66. *Given a communicating class $C \subset V$ and its lift, Ω_C: Ω_C is isolated.*

Proof. Let B be a neighborhood of Ω_C, and let $x \in \Omega$ such that $\omega(x), \alpha(x) \subset B$. For any $p \in B$ there exists some $q \in \Omega_C$ such that $d(p,q) < \varepsilon$. Since $\alpha(x), \omega(x) \subset B$, there exist $i, j \in \mathbb{Z}$ such that for all $t \leq i$ $\pi_t(x) \in C$ and for all $t \geq j$ $\pi_t(x) \in C$. Now consider any k such that $i < k < j$. Since x is an admissible sequence there exists $\gamma \in \mathcal{P}$ with $\gamma_1 = \pi_i(x)$, $\gamma_F = \pi_F(x)$ and $\gamma_m = \pi_k(x)$ where $\gamma = (\gamma_1, ..., \gamma_m, ..., \gamma_f)$. Since $\pi_i(x), \pi_j(x) \in C$ there exists $\gamma' \in \mathcal{P}$ with $\gamma'_1 = \pi_j(x)$ and $\gamma'_F = \pi_i(x)$. So we can construct paths $(\gamma_m, ..., \gamma_F), (\gamma_m, ..., \gamma'_1, ..., \gamma'_F), (\gamma'_1, ..., \gamma'_F, \gamma_2, ..., \gamma_m) \in \mathcal{P}$ to be between $\pi_i(x), \pi_j(x), \pi_k(x)$. So $\pi_k(x) \in C$. Since k was arbitrary $x \in \Omega_C$. □

In Chapter 2 we also introduced the concept of Poincaré recurrence. We now examine this idea within the context of (Ω, Φ).

Proposition 67. *If $x \in \Omega$ is Poincaré recurrent, then $x \in \Omega_C$ for some communicating class C.*

Proof. By Lemma 52, $\alpha(x)$ and $\omega(x)$ are each contained in a particular communicating class. Note that if x is Poincaré recurrent, $\omega(x) \cap \alpha(x) \neq \emptyset$, as $x \in \omega(x) \cap \alpha(x)$. Thus, if x is Poincaré recurrent, then $\alpha(x), \omega(x) \subset \Omega_C$ for one particular communicating class C. This implies by Lemma 66 (the isolation of lifts of communicating classes) that $x \in \Omega_C$. □

However, the above does not imply that all points in the lift of a communicating class are Poincaré recurrent; in fact, the following example shows that there can be points within lifts of communicating classes that are not Poincaré recurrent.

Example 68. *Let G be the complete graph on 2 vertices (labeled 1 and 2). Then since G is a single communicating class, Ω is a lift of a communicating class. Consider the sequence*

$$x = (\ldots, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, \ldots).$$

Then $\alpha(x) = \{(\ldots, 1, 1, 1, 1, 1, 1, \ldots)\},$, while $\omega(x) = \{(\ldots, 2, 2, 2, 2, 2, 2, \ldots)\}$. Thus, $\alpha(x) \cap \omega(x) = \emptyset$, and thus x can not be Poincaré recurrent.

We can, however, categorize to an extent the Poincaré recurrent points of (Ω, Φ), and determine how common they are within lifts of communicating classes.

Proposition 69. *Periodic points are Poincaré recurrent.*

Proof. If x is periodic with period τ, then

$$x = \lim_{n\to\infty} \phi_{n\tau}(x) = \lim_{n\to-\infty} \phi_{n\tau}(x).$$

Thus, $x \in \omega(x)$ and $x \in \alpha(x)$, and is therefore Poincaré recurrent. □

Corollary 70. *Poincaré recurrent points are dense within lifts of communicating classes.*

Proof. This follows directly from the above proposition and Lemma 47. □

So while not every point in Ω_C is Poincaré recurrent, every point can be approximated by a sequence of Poincaré recurrent points. It should thus be noted that the set of Poincaré recurrent points is not closed.

Additionally we discussed the idea of nonwandering sets of a dynamical system. All of these constructions are tools used to understand the dynamics on (Ω, Φ), and the more accurately we can identify them, the better our understanding of the system is. Thus, we examine the nonwandering set of (Ω, Φ).

Proposition 71. *Given a directed graph G, the nonwandering set of ϕ on Ω is the union of the lifts of communicating classes of G.*

Proof. To start, we show that the nonwandering set is contained in the union of the lifts of communicating classes. Let x be a nonwandering point. Then, for all $\varepsilon > 0$ there exists a $y \in B(x, \varepsilon)$ and $T > 1$ such that $\phi_T(y) \in B(x, \varepsilon)$. If we let $\varepsilon < 1$, then we note that $\phi_T(y)_0 = y_0 = x_0$ (that is, the three zeroth entries of the three sequences must be equal). Note further that $\phi_T(y)_0 = y_T = x_0$. Since $T > 1$, this implies that there must be a path from x_0 back to x_0 in G (given by the sequence $\{y_0, y_1, \ldots, y_T\}$), which, by Lemma 9, implies that x_0 must be contained within a communicating class of G. By a similar method, letting $\varepsilon < \frac{1}{4^{|n|}}$, it can be demonstrated that every x_n must be contained within a communicating class. By maximality of communicating classes, they must all be contained within the same communicating class (which we will denote as C), and thus, by definition of a lift of communicating class, x is contained in Ω_C.

To show that the union of the lifts of communicating classes is contained within the nonwandering set, we begin by demonstrating that all periodic points are nonwandering points. Let p be a periodic point. Then there exists $T > 1$ such that $\phi_T(p) = p$. Thus, for all open neighborhoods U of p, $\phi_T(p) \in U$, and thus p is nonwandering. Note that by Proposition 23, the nonwandering set must then contain the closure of the set of periodic points. By Lemma 47, the closure of the set of periodic points is equal to the union of lifts of communicating classes. Thus, the nonwandering set must contain the union of the lifts of communicating classes. □

Thus, is appears that the lifts of communicating classes are where any form of recurrence occur within Ω. They are exactly the chain recurrent set and nonwandering set, and they contain all of the Poincaré recurrent points. In addition, they hold all the α, ω-limit sets. We have also noted their invariance and isolation. It's thus clear that they are of tantamount importance to the structure of the system (Ω, Φ). One could think of them as the skeleton of the system upon which everything else is built. It would be nice to sum all of this up with one term. Such a term is introduced now.

Definition 72. *A Morse Decomposition of a flow on a compact metric space is a finite collection $\{\mathcal{M}_i, i = 1, ..., n\}$ of nonvoid, pairwise disjoint, and compact isolated invariant sets such that:*

- *(i) For all $x \in \Omega$ one has $\omega(x), \alpha(x) \subset \bigcup_{i=1}^{n} \mathcal{M}_i$.*

- *(ii) Suppose there are $\mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_l$ and $x_1, ..., x_l \in \Omega \backslash \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{i-1}$ and $\omega(x_i) \subset \mathcal{M}_i$, for $i = 1, ..., l$, and $\mathcal{M}_0 \neq \mathcal{M}_l$. The elements of a Morse Decomposition are called Morse Sets.*

(3)

Remark 73. *A Morse Decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_{n'}\}$ is called $finer$ than a Morse Decomposition $\{\mathcal{M}'_1, ..., \mathcal{M}'_n\}$, if for all $j \in \{1, ..., n\}$ there is $i \in \{1, ..., n'\}$ with $\mathcal{M}_i \subset \mathcal{M}'_j$, and the containment is strict for at least one j. In this case, the Morse Decomposition $\{\mathcal{M}'_1, \ldots, \mathcal{M}'_n\}$ is said to be coarser than $\{\mathcal{M}_1, \ldots, \mathcal{M}_{n'}\}$.*

One should note that, with any compact dynamical system (M, Φ), it is easy to find at least one example of a Morse Decomposition. The whole set M considered as a single Morse set rather trivially satisfies the definition of a Morse Decomposition - it is the coarsest . Thus, we strive to find finer Morse Decompositions. The reader can probably guess that a candidate for a Morse Decomposition for (Ω, Φ) is the collection of lifts of communicating classes.

The next lemma is used to show that the lifts of communicating classes can not be divided into subsets that serve as a finer Morse Decomposition.

Lemma 74. *Given nonempty and invariant sets $U_1, U_2 \subset \Omega_C$, where $U_1 \cap U_2 = \emptyset$ and C is any communicating class in the N-graph G, there exists a cycle between U_1, U_2.*

Proof. For any $x \in U_1$ and $\varepsilon > 0$ there exist $B(x, \varepsilon)$ which, by Lemma 40, contains the set $\{q \in \Omega \mid q_i = x_i \text{ for } -N \leq i \leq N \text{ and some } N \in \mathbb{N}\}$. Therefore we can construct a point $r \in B(x, \varepsilon)$ such that $r_i = x_i$ for all $i \leq -N$ and $r_i = z_i$ with $z \in U_2$, for all $i \geq N + k$ where $k \in \mathbb{N}$ is the length of the minimal path between $\pi_N(x)$ and $\pi_{k+1}(z)$. We know this minimal path exists because for all $i \in \mathbb{Z}$, $\pi_i(x), \pi_i(z) \in C$ a communicating class. Then since sequences of shifts of r with $-t$ will converge to shifts of $x \in U_1$, $\alpha(r) \subset U_1$ because U_1 is invariant. Similarly $\omega(r) \subset U_2$ because U_2 is invariant.

With a similar argument a y can be found in the neighborhood of U_2 such that $\alpha(y) \subset U_2$ and $\omega(y) \subset U_1$. Therefore a cycle exist between the two subsets of Ω_C. □

We finally end this section with the proof of the fact that the lifts of communicating classes form a finest Morse Decomposition for (Ω, Φ).

Theorem 75. *Given an N-graph, G, and associated shift space $\Omega \subset \Upsilon$, the lifts of the communicating classes in G, represented as elements of $\mho_S$, are Morse Sets in Ω which form a finest Morse Decomposition for Ω.*

Proof. Clearly the set of communicating classes of G is finite because G is finite, so $\mho_S$ has finitely many elements. Also, by the definition of communicating classes, elements of $\mho_S$ are nonvoid. Then, from Lemmas 34, 66, 46, and 32 that they are also disjoint, isolated, compact, and invariant. In addition they fulfill the requirements of (i) and (ii) by Lemmas 52 and 51 respectively. Thus the lifts of communicating classes of an N-graph, G, form a Morse Decomposition for Ω associated with G.

To show that $\mho_S$ is the finest Morse Decomposition we suppose there exists some finer Morse Decomposition $\mathcal{M} = \{\mathcal{M}_1, ..., \mathcal{M}_{n'}\}$. Then by definition for all $j \in \{1, ..., n\}$, where n is the number of communicating classes in G, there is $i \in \{1, ..., n'\}$ with $\mathcal{M}_i \subset \Omega_{C_j}$. If $n = n'$, then for each Ω_{C_j} there exists exactly one $\mathcal{M}_i$ such that $\mathcal{M}_i \subset \Omega_{C_j}$. If $\mathcal{M}_i = \Omega_{C_j}$ for all i, j then $\mho_S = \mathcal{M}$, so assume there are some i, j where the strict subset applies. Then pick $x' \in \Omega_{C_j}$ such that $x' \notin \mathcal{M}_i$. By Lemma 55 there exists $x^j \in \Omega_{C_j}$ such that for all $x \in \Omega_{C_j}$, $x \in \omega(x^j)$. Since $x' \notin \mathcal{M}_i$ and $x' \in \omega(x^j)$, $\omega(x^j) \not\subset \mathcal{M}_i$, and because the lifts of communicating classes are disjoint $\omega(x^j) \not\subset \bigcup_{i=1}^{n'} \mathcal{M}_i$. Thus $\mathcal{M}$ is not a Morse Decomposition because it does not follow requirement (i). If $n < n'$ then for some Ω_{C_j} there exist $\mathcal{M}_i, \mathcal{M}_k \subset \Omega_{C_j}$. Since $\mathcal{M}_i, \mathcal{M}_k$ must be invariant to be Morse Sets, from Lemma 74 we know that there are cycles between $\mathcal{M}_i, \mathcal{M}_k$, so $\mathcal{M}$ does not follow requirement (ii). Thus $\mathcal{M}$ is not a Morse Decomposition. Therefore, because there exists no finer Morse Decomposition, $\mho_S$ is the finest Morse Decomposition. $\square$

Thus, we in a way have a very clear understanding of the dynamics on (Ω, Φ). We understand that within communicating classes we either encounter a single periodic orbit or chaotic behavior and topological mixing, thus ensuring that lifts of communicating classes can not be broken up into smaller Morse Sets. Additionally, outside of the lifts of communicating classes, there is convergence both towards and away from communicating classes. The orbit of any point x that is not contained within the lift of a communicating class will converge forward toward one lift, and away from another. Thus there is a rather nice “unidirectional” flow outside the lifts of communicating classes. With this system now understood, we move on to examining what happens when we lift this system to a skew product flow when we consider hybrid systems.

CHAPTER 4. HYBRID DYNAMICAL SYSTEMS

Up until this point, we have examined only a discrete time dynamical system; namely, (Ω, Φ) which takes integer time values. However, our ultimate interest lies in continuous time dynamical systems, where the only discrete behavior is seen from the switching between different dynamical systems on the same space, which happens instantaneously and at regular time intervals. This behavior was isolated and examined on its own in Chapter 3. Now we turn our attention to a flow on the space $M \times \Delta$, where M is a compact subset of $\mathbb{R}^n$, and Δ, defined more rigorously below, is a "continuization" of Ω; that is, it is a system that is isometrically isomorphic to Ω. In this section, we first define and examine behavior on Δ. We then look at the system on $M \times \Delta$, where we will show that this system is a skew product flow and then examine some limit and recurrence concepts within this context.

4.1 $\bar{\Delta}$ and Δ

Note that in (1), the flow on Ω is a discrete time dynamical system. However, to insert this behavior into another dynamical systems to create a hybrid system, it requires that we extend this system to a continuous time dynamical system. The obvious extension of a sequence into a function on $\mathbb{R}$ is a piecewise constant function.

Definition 76. *Given an N-graph $G = (V, E)$ and a fixed $h > 0$, let*

$$\bar{\Lambda} = \left\{ x : \mathbb{R} \to V \;\middle|\; \begin{array}{c} \{x(ih)\}_{i \in \mathbb{Z}} \in \Upsilon \\ x \textit{ is piecewise constant on } [nh, (n+1)h) \, \forall \, n \in \mathbb{Z} \end{array} \right\}$$

and

$$\Lambda = \{x(\cdot + t) | x \in \Lambda, t \in \mathbb{R}\}.$$

Definition 77. *Let*

$$\bar{\Delta} = \left\{ x : \mathbb{R} \to V \;\middle|\; \begin{array}{c} \{x(ih)\}_{i\in\mathbb{Z}} \in \Omega \\ x \textit{ is piecewise constant on } [nh,(n+1)h)\,\forall\, n \in \mathbb{Z} \end{array} \right\}$$

and

$$\Delta = \{x(\cdot + t) | x \in \Delta, t \in \mathbb{R}\}.$$

In other words, Λ and Δ are the sets of functions that result from translating the functions in $\bar{\Lambda}$ and $\bar{\Delta}$, respectively, by some $t \in \mathbb{R}$. We allow for all horizontal translations of functions in $\bar{\Lambda}$ and $\bar{\Delta}$ in order for the spaces to be closed under shifts by t for all $t \in \mathbb{R}$.

The next definition adapts the shift operator to continuous time by taking functions in $\bar{\Delta}$ as a generalization of bi-infinite sequences in Ω.

Definition 78. *Let*

$$\begin{aligned} \psi : \mathbb{R} \times \Delta &\to \Delta \\ (t, x(\cdot)) &\mapsto x(\cdot + t) \end{aligned}$$

Note that ψ satisfies the flow property:

$$\psi(s+t, x(k)) = x(k+s+t) = x((k+t)+s) = \psi(s, x(k+t)) = \psi(s, \psi(t, x(k))).$$

We now use the following to establish a metric on the set of functions Δ.

Definition 79. *Define the function*

$$\begin{aligned} f : \Delta \times \Delta \times \mathbb{Z} &\to \mathbb{R} \\ (x, y, i) &\mapsto \tfrac{1}{h} \int_{ih}^{(i+1)h} \delta(x, y, t)dt \end{aligned}$$

where

$$\delta(x, y, t) = \begin{cases} 1 & \textit{if } x(t) \neq y(t) \\ 0 & \textit{if } x(t) = y(t) \end{cases}$$

Theorem 80. *The function*

$$\begin{aligned} d : \Delta \times \Delta &\to \mathbb{R} \\ (x, y) &\mapsto \sum_{i=-\infty}^{\infty} \frac{f(x, y, i)}{4^{|i|}} \end{aligned}$$

is a metric on $\bar{\Delta}$.

Proof.

1. (Non-negativity) $f(x,x,i) = 0$, for all $i \in \mathbb{Z}$. Therefore, $d(x,x) = 0$. For $x \neq y$, $f(x,y,i) \neq 0$ for at least one $i \in \mathbb{Z}$, and $f(x,y,i) \geq 0$ for all $i \in \mathbb{Z}$. Therefore, $d(x,y) > 0$ for all $x \neq y$.

2. (Symmetry) Clearly, $\delta(x,y,t) = \delta(y,x,t)$ for all $t \in \mathbb{R}, x, y \in \Delta$. So, $f(x,y,i) = f(y,x,i)$, $d(x,y) = d(y,x)$.

3. (Triangle inequality) Choose $x,y,z \in \Delta$. If $x = z$, then as $d(x,z) = 0$ and d is nonnegative, then clearly for all $y \in \Delta$, $d(x,z) \leq d(x,y) + d(y,z)$.

   If $x \neq z$, then there exists $t \in \mathbb{R}$ such that $x(t) \neq z(t)$. If $x(t) \neq z(t)$, then either $x(t) \neq y(t)$ or $y(t) \neq z(t)$. Therefore, $\delta(x,z,t) = 1$ implies that $\delta(x,y,t) = 1$ and/or $\delta(y,z,t) = 1$, so $f(x,z,i) \leq f(x,y,i) + f(y,z,i)$. Since d is a linear combination of f's, $d(x,z) \leq d(x,y) + d(y,z)$.

□

Because we would like to consider Δ as essentially the continuous version of Ω, it helps to establish a relationship between them. We show below that the spaces Ω and $\bar{\Delta}$ (the version that does not contain all real time shifts of the functions) are isometrically isomorphic, and are in fact topologically conjugate as well.

Proposition 81. *The mapping $\sigma : \Omega \to \bar{\Delta}$ where $x \mapsto x(t)$ where $x(i) = x_i$ for all $i \in \mathbb{Z}$ is an isometric isomorphism (bijection).*

Proof. By the construction of $\bar{\Delta}$, σ is clearly bijective.

To show that σ is an isometry, it suffices to show that $f(x,y,i) = \bar{f}(x_i, y_i)$, where

$$\bar{f}(x_i, y_i) = \begin{cases} 1 & \text{if } x(i) \neq y(i) \\ 0 & \text{if } x(i) = y(i) \end{cases}$$

since the bi-infinite sums for d and $\bar{d}$ are identical. Note that

$$f(x,y,i) = \frac{1}{h} \int_{ih}^{(i+1)h} dt = 1 = \bar{f}(x_i, y_i)$$

for $x(i) \neq y(i)$.

$$f(x,y,i) = \frac{1}{h}\int_{ih}^{(i+1)h} 0 \cdot dt = 0 = \bar{f}(x_i, y_i)$$

for $x(i) = y(i)$. So indeed, $f(x,y,i) = \bar{f}(x_i, y_i)$, and $d(x,y) = \bar{d}(\{x_i\}, \{y_i\})$.

□

Before we show the topological conjugacy of Ω and $\bar{\Delta}$, we first introduce the concept.

Definition 82. *A flow φ on a space X is said to be topologically conjugate to a flow ψ on a space Y if there is a homeomorphism $h : X \to Y$ such that $\psi(h(x), t) = h(\phi(x,t))$.*

Systems that are topologically conjugate can be shown to exhibit the same properties. For instance, given the dynamics on Ω, we then are already familiar with the dynamics on $\bar{\Delta}$. Thus, these two systems are essentially equivalent. It is for this reason that we demonstrate this now.

Theorem 83. *(Ω, Φ) and $(\bar{\Delta}, \psi|_{\mathbb{Z}h})$ are topologically conjugate.*

Proof. Let $\sigma : \Omega \to \bar{\Delta}$ be defined as in Proposition 81. Because σ has been shown to be a bijective isometry, it is automatically a homeomorphism. We thus claim that σ is the homeomorphism that satisfies the definition of topologically conjugate given by Definition 82; that is,

$$\psi(\sigma(x), nh) = \sigma(\Phi(x, nh))$$

for all $n \in \mathbb{Z}$. Let $i \in \mathbb{Z}$; if we can show that $\sigma(\Phi(x,n))(ih) = \psi(\sigma(x), nh)_{ih}$, then the proof is complete. Notice that the ith component of $\Phi(x,n)$ is x_{i+n}. Thus, $\sigma(\Phi(x,n))(ih) = x_{i+n}$. Similarly, $\sigma(x)((i+n)) = x_{i+n}$, and thus $\psi(\sigma(x), nh)_{ih}$.. Therefore, the result holds, and (Ω, Φ) and $(\bar{\Delta}, \psi|_{\mathbb{Z}h})$ are topologically conjugate. □

Because of the topological conjugacy, $\bar{\Delta}$ inherits a lot of properties from Ω, many of which we go through here. Notice that many of the proofs are very similar, and many rely on Ω having the same properties. However, we can not rely on topological conjugacy alone because the space that we are really interested in is Δ, not $\bar{\Delta}$, which is a much bigger space containing Δ as a framework, but is not topologically conjugate to Ω. For instance, below we discuss the continuity of the shift.

Lemma 84. *ψ_t is continuous for all $t \in \mathbb{R}$.*

Proof. Given $x, y \in \Delta$, we need to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < \delta \Rightarrow d(\psi_t(x), \psi_t(y)) < \epsilon.$$

Given any $\epsilon > 0$, take $\delta = \frac{\epsilon}{4^n}$, where $n = \lceil |\frac{t}{h}| \rceil$, the least integer greater than the absolute value of $\frac{t}{h}$. It is useful to rewrite $d(x,y)$ in the form

$$d(x,y) = \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{4^{\left|\lfloor \frac{t'}{h} \rfloor\right|}} \delta(x,y,t')dt'$$

where δ is as defined above. Given this, we can write

$$d(\psi_t(x), \psi_t(y)) = \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{4^{\left|\lfloor \frac{(t+t')}{h} \rfloor\right|}} \delta(x,y,t')dt'$$

And,

$$\frac{1}{4^{\left|\lfloor \frac{(t+t')}{h} \rfloor\right|}} \leq 4^{\lceil |\frac{t}{h}| \rceil} \frac{1}{4^{\left|\lfloor \frac{t'}{h} \rfloor\right|}}$$

So,

$$\begin{aligned}
d(\psi_t(x), \psi_t(y)) &= \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{4^{\left|\lfloor \frac{(t+t')}{h} \rfloor\right|}} \delta(x,y,t')dt' \\
&\leq 4^{\lceil |\frac{t}{h}| \rceil} \frac{1}{h}\int_{-\infty}^{\infty} \frac{1}{4^{\left|\lfloor \frac{t'}{h} \rfloor\right|}} \delta(x,y,t')dt' \\
&= 4^{\lceil |\frac{t}{h}| \rceil} d(x,y) \\
&< 4^{\lceil |\frac{t}{h}| \rceil} \delta \\
&= \epsilon
\end{aligned}$$

□

Similarly to Ω, we have compactness of Δ. (We already know $\bar{\Delta}$ is compact since is it homeomorphic to Ω.)

Lemma 85. *Δ is compact.*

Proof. We will show that given any sequence $\{x^n\}, n \in \mathbb{N}$ of functions $x^n \in \Delta$, there exists a subseqence converging to some $x \in \Delta$. To do this, we consider the space Δ to be the product of a circle of length h with the set of allowable bi-infinite sequences, $S^1 \times \Omega$, where $S^1 \equiv \mathbb{R}$ mod h; that is, $\Delta \sim S^1 \times \overline{\Delta} \sim \Omega \times S^1$ (not with the product topology, however). An element x^n of Δ identifies with an element of $S^1 \times \Omega$ by taking $y^n \in \Omega$ to be the sequence of constant values of x^n, with $x^n(0) \equiv y_0^n, x^n(h) \equiv y_1^n$, etc., and taking $\tau \in [0, h)$ to be the unique offset so that $x^n(t - \tau) \in \Delta$.

S^1 is compact. Therefore, given the sequence $\{x^n\} \in S^1 \times \Omega$, there exists a subsequence $\{x^{n_k}\}, k \in \mathbb{N}$ for which the offsets $\{\tau^{n_k}\}$ converge to a value in $[0, h)$. Therefore, there exists a convergent subsequence $\{\tau^{n_k}\}$ of $\{\tau^n\}$.

For this subsequence $\{x^{n_k}\}$, we want to show that there exists a subsequence $\{x^{n_{k_j}}\}, j \in \mathbb{N}$ such that the bi-infinite sequences $\{y^{n_{k_j}}\}$ converge. We do this inductively, beginning with the subsequence $\{y_0^{n_{k_j}}\}$. We know that $\{y_0^{n_k}\}$ is an infinite sequence of finitely many values, since the state space S is finite. Therefore, by the pigeonhole principle, there is one value that is repeated infinitely many times. Take $\{y_0^{n_{k_j}}\}$ to be this value, s_0, so that $x^{n_{k_j}}(0) = s_0$ for all $j \in \mathbb{N}$.

Now, we induct. Given a subsequence of $\{x^{n_k}\}$ that converges at $t = 0, h, -h, 2h, -2h, ...,$ $mh, -mh$, we deduce that there must be a subsequence of this subsequence with one value $x^{n_k}((m+1)h) = s_{(m+1)h}$ repeated infinitely many times, and likewise for $x^{n_k}(-(m+1)h) = s_{-(m+1)h}$. In this manner, we get an infinite subsequence $\{y^{n_{k_j}}\}$, hence $\{x^{n_{k_j}}\}$, converging to a function that is piecewise constant on $[\tau + nh, \tau + (n+1)h), n \in \mathbb{Z}, \tau \in [0, h]$, with values in S.

Finally, we have to show closure. That is, we need to show that transitions $x(mh) \to x((m+1)h)$ in our limit function are allowable. Otherwise, all we would have shown is compactness of $\bar{\Lambda}$, rather than compactness of Δ. Suppose that $x \notin \Delta$. Then, there exists some $m \in \mathbb{Z}$ such that the transition $x(mh) \to x((m+1)h)$ is not allowed. But, since $\{x^{n_{k_j}}\}$ converges to x, we can take N large enough so that $j > N \Rightarrow x^{n_{k_j}}(mh) = x(mh), x^{n_{k_j}}((m+1)h) = x((m+1)h)$. And, $x^{n_{k_j}} \in \Delta$, so the transition $x^{n_{k_j}}(mh) \to x^{n_{k_j}}((m+1)h)$ must be allowable. This is a contradiction. Therefore, $\{x^{n_{k_j}}\} \to x \in \Delta$.

$\square$

4.1.1 Morse Sets and Topological Chaos

For the following definitions and Proposition 89, taken from (3), let X be a compact metric space with an associated flow Φ. These definitions have already been introduced in Chapter 3; we merely restate them here to remind the reader.

Definition 86. *A set $K \subseteq X$ is called invariant if $\Phi(t,x) \in K$ for all $x \in K, t \in \mathbb{R}$.*

Definition 87. *A set $K \subseteq X$ is called isolated if there exists a neighborhood N of K (i.e. a set N with $K \subset int\ N$) such that $\Phi(t,x) \in N$ for all $t \in \mathbb{R}$ implies $x \in K$.*

Definition 88. *A Morse Decomposition on X is a finite collection $\{\mathcal{M}_i, i = 1, ..., n\}$ of non-void, pairwise disjoint, invariant, isolated, compact sets such that*

1. *For all $x \in X, \omega(x), \alpha(x) \subseteq \bigcup_{i=1}^{n} \mathcal{M}_i$.*
2. *If there exist $\mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_l$ and $x_1, ...x_l \in X \setminus \bigcup_{i=1}^{n} \mathcal{M}_i$ with $\alpha(x_j) \subseteq \mathcal{M}_{j-1}$ and $\omega(x_j) \subseteq \mathcal{M}_j$ for $j = 1, ..., l$, then $\mathcal{M}_0 \neq \mathcal{M}_l$. This condition is equivalent to to the statement that there are no cycles between the sets of the Morse decomposition.*

The sets $\mathcal{M}_i$ above are called Morse sets.

Proposition 89. *The relation $\preceq$ given by*

$$\mathcal{M}_i \preceq \mathcal{M}_k \text{ if there are } \mathcal{M}_i, \mathcal{M}_{j_1}, ..., \mathcal{M}_{j_l} = \mathcal{M}_k \text{ and } x_1, ..., x_l \in X$$

$$\text{with } \alpha(x_m) \subseteq \mathcal{M}_{j_{m-1}} \text{ and } \omega(x_m) \subseteq \mathcal{M}_{j_m} \text{ for } m = 1, ..., l.$$

is an order (satisfying reflexivity, transitivity, and antisymmetry) on the Morse sets $\mathcal{M}_j$ of a Morse decomposition.

The proof of this proposition can be found in (3).

Also, similar to the concept of lifts of communicating classes in Ω, we have lifts of communicating classes in Δ.

Definition 90. *The lift $\Delta_C \subseteq \Delta$ of a communicating class C is defined by*

$$\Delta_C \equiv \{f \in \bar{\Delta} | f(t) \in C \text{ for all } t \in \mathbb{R}\}$$

$\bar{\Delta_C}$ is defined as

$$\bar{\Delta}_C \equiv \Delta_C \cap \bar{\Delta}.$$

That is to say, Δ_C contains all real time shifts of functions in $\overline{\Delta_C}$.

In other words, given a communicating class C, Δ_C is the set of all functions $f \in \Delta$ whose ranges are contained in C.

Unsurprisingly, these lifts display many of the same qualities as those exhibited in Ω, the first being that they form a Morse Decomposition for (Δ, ψ).

Theorem 91. *The lifts of the communicating classes Δ_C are Morse sets for the dynamical system ψ.*

Proof. We check each of the conditions in turn.

1. **Non-void** Since the empty set is not a communicating class, the lift of any communicating class must be non-empty.

2. **Pairwise disjoint** Suppose that there exists $f \in \Delta_C, \Delta_{C'}$ with $C \neq C'$. Then, $f(0) \in C, C'$. But, by the maximality of communicating classes, $f(0) \in C, C'$ implies $C = C'$. So, $\Delta_C = \Delta_{C'}$.

3. **Invariant** By construction of Δ, $\psi(t, f) \in \Delta$ for all $t \in \mathbb{R}, f \in \Delta$. And, if $f(s) \in C$ for all $s \in \mathbb{R}$, then $\psi(t, f)(s) = f(t + s) \in C$. So, $\psi(t, f) \in \Delta_C$ for all $t \in \mathbb{R}, f \in \Delta_C$.

4. **Isolated** Pick $\epsilon = 1/4$. Suppose that there exists $g \notin \Delta_C$ such that for some $f \in \Delta_C$, $d(g, f) < \epsilon$. Since $g \notin \Delta_C$, there exists t_0 such that $g(t_0) \notin C$. Let $g' = \psi(-t_0, g)$, so that $g'(0) \notin C$. But then, g' differs from any function in Δ_C on at least some interval of length h containing 0. The distance, therefore, between g' and any function in Δ_C must be greater than $1/4$. So, given any $g \notin \Delta_C$ but within $1/4$ of Δ_C, there exists t_0 such that $d(\psi(t_0, g), f') > \frac{1}{4}$ for any $f' \in \Delta_C$. Hence, Δ_C is isolated.

5. **Compact** By an argument similar to that for compactness of Δ and by compactness of Ω_C, Δ_C is compact.

6. **No cycles** Again, this is similar to the corresponding proof in (1). Suppose that there exist $f, g \in \bar{\Delta}$ such that $\alpha(f) \subseteq \bar{\Delta}_C$, $\alpha(g) \subseteq \bar{\Delta}_{C'}$ and $\omega(g) \subseteq \bar{\Delta}_C$, $\omega(f) \subseteq \bar{\Delta}_{C'}$. Then, since all the transitions in f, g must be allowable, there must exist an admissible path from C to C' as well as one from C' to C. But, this contradicts maximality of communicating classes. So, no such cycle exists.

□

It turns out that the flow within each of the lifts of communicating classes is topologically transitive as well, just like as in Ω. We first restate the definition of topological transitivity below.

Definition 92. *A flow on a metric space X is called topologically transitive if there exists $x \in X$ such that $\omega(x) = X$.*

Lemma 93. *Given any communicating class C, there exists $f^* \in \Delta_C$ such that $\omega(f^*) = \Delta_C$ (i.e. ψ is topologically transitive on lifts of communicating classes).*

Proof. It has been shown in (1) that for the discrete system, there exists $x^* \in \Omega_C$ such that $\omega(x^*) = \Omega_C$. This proof is also state in Chapter 2. By the correspondence σ as defined in Proposition 81 between sequences in Ω and functions in Δ, there exists $f^* \in \Delta_C$ given by

$$f^*(nh) = x^*_n, n \in \mathbb{Z}$$

such that $\Delta_C \subseteq \omega(f^*)$. And, since $\bar{\Delta}_C$ is given by the shifts $\psi(t, \Delta_C)$, it is clear that $\Delta_C \subseteq \omega(f^*)$. Δ_C is invariant by Theorem 91, so $\omega(f^*) \subseteq \Delta_C$, and $\omega(f^*) = \Delta_C$. □

Since the ω-limit sets of a point on a compact space are connected, we get the following corollary.

Corollary 94. *Δ_C is connected.*

Once again, just as in Ω_C, we see that these points are dense in Δ_C for any communicating class C.

Proposition 95. *The set of all functions f^* satisfying $\omega(f^*) = \Delta_C$ is dense in Δ_C*

Proof. Given $f \in \Delta_C$, there exists f^* such that $f \in \omega(f^*)$, by Lemma 93. Therefore, given $\epsilon > 0$, there exists $t \in \mathbb{R}$ such that $d(\psi(t, f^*), f) < \epsilon$. $\omega(f^*) = \Delta_C$ implies $\omega(\psi(t, f^*)) = \Delta_C$. So, for any $f \in \Delta_C$ and any $\epsilon > 0$, there exists a function $\psi(t, f^*) \in \Delta_C$ with $d(\psi(t, f^*), f) < \epsilon$ and $\omega(\psi(t, f^*)) = \Delta_C$. □

The above propositions serve to show that the lifts of communicating classes Δ_C can not be broken up into smaller invariant components, which serves useful in showing that this Morse Decomposition given by the lifts of communicating classes is in fact the finest Morse Decomposition that exists on Δ, ψ.

Definition 96. *A Morse Decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_n\}$ is called finer than a Morse Decomposition $\{\mathcal{M}'_1, ..., \mathcal{M}'_l\}$ if for all $j \in \{1, ..., l\}$ there exists $i \in \{1, ..., n\}$ such that $\mathcal{M}_i \subseteq \mathcal{M}'_j$, where containment is proper for at least one j.*

Theorem 97. *The lifts of the communicating classes Δ_C form a finest Morse decomposition on Δ.*

Proof. Suppose there exists a finer Morse decomposition, $\mathcal{M}_\infty, \ldots, \mathcal{M}_k$. Then, for some communicating class C, there exists a Morse set $\mathcal{M}_i \subsetneq \Delta_C$, a proper containment. By the definition of a Morse set, $\mathcal{M}_i$ must contain the ω-limit sets of Δ_C. However, by Lemma 93, there exists $f^* \in \Delta_C$ such that $\omega(f^*) = \Delta_C$. Therefore, $\Delta_C \subseteq \mathcal{M}_i$, so $\mathcal{M}_i$ is not a proper subset of Δ_C. Thus, no finer Morse decomposition exists.

□

Lastly, as one would expect, the behavior of the flow on the lifts of communicating classes exhibits chaotic behavior and sensitive dependence on initial conditions as well, just as in Ω.

Definition 98. *A flow Φ on a metric space X has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood B of x, there exists $y \in B$ and $t > 0$ such that $d(\Phi_t(x), \Phi_t(y)) > \delta$.*

Definition 99. *A flow on a metric space X is chaotic if it has sensitive dependence on initial conditions, density of periodic points, and is topologically transitive.*

Lemma 100. *Consider a graph G consisting of a single communicating class C for which the out-degree of at least one vertex is at least two. Then, ψ on Δ has sensitive dependence on initial conditions.*

Proof. Take $\delta = \frac{1}{2}$. Given $x \in \Delta$, we construct a function $y \in \Delta$ such that x and y are discontinuous at the same times mod h. Given $\varepsilon > 0$, take N large enough so that

$$\sum_{i=-\infty}^{-N} \frac{1}{4^{|i|}} + \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon.$$

Thus, taking $x(t) = y(t)$ on $t \in [-Nh, Nh]$ ensures that $d(x, y) < \varepsilon$. Now, we just need to show that there exists $m > N \in \mathbb{R}$ and $y \in \Delta$ so that $y(t) \neq x(t)$ for all $t \in [mh, (m+1)h)$. This would imply that $d(\psi(mh, x), \psi(mh, y)) \geq 1 > \delta$. To show that such an m exists, let γ_1 denote the vertex with out-degree greater than one. If there does not exist a $t > Nh$ such that $x(t) = \gamma_1$, then given $x(Nh) = \gamma_N$, let $y(t), t > Nh$ follow a path from γ_N to γ_1. Such a path must exist since G consists of a single communicating class, so there exists a path between any two vertices in G. Thus, we would have $x(t_0 + t) \neq \gamma_1, y(t_0 + t) = \gamma_1$ for some $t_0 > Nh, t \in [0, h)$, so we can take $m = \frac{t_0}{h}$. If there does exist a $t > Nh$ such that $x(t) = \gamma_1$, then define $\gamma_2 = x(t + h)$, and take t_1 so that $x(t_1 + t) = \gamma_1$ for $t \in [0, h)$. Since the out-degree of γ_1 is greater than one, there exists an edge from γ_1 to some other vertex, γ_3 (note that it is possible that either $\gamma_1 = \gamma_2$ or $\gamma_1 = \gamma_3$, but not that $\gamma_2 = \gamma_3$). Set $y(t_1 + h) = \gamma_3$, and take $m = \frac{t_1+h}{h}$.

□

Note that Lemma 100 can also be proven by noticing that density of periodic points and topological transitivity imply sensitivity of initial conditions, and thus chaos.

Considering together Lemmas 93 and 100 yields the following result.

Theorem 101. *Consider a graph consisting of a single communicating class for which the out-degree of at least one vertex is greater than one. Then, ψ is chaotic on Δ.*

Corollary 102. *ψ is chaotic on lifts of communicating classes (where the out-degree of at least one vertex is greater than one). If the out degree of every vertex is exactly one, then the lift of the communicating class is a single periodic orbit.*

Finally, with confirmation that behavior in Δ mirrors that of Ω as explained in Chapter 3, we move on to considering the hybrid system defined on $M \times \Delta$.

4.2 Deterministic Hybrid Systems

Now that the behavior on Δ has been determined given a natural number n and an N-graph on n vertices, we consider the action of a function $f \in \Delta$ on a set of n dynamical systems.

Consider an N-graph G with n vertices. Take a collection of n dynamical systems $\{\phi_1, ..., \phi_n\}$ on a compact space $M \subset \mathbb{R}^d$, where each vertex of G corresponds to one dynamical system ϕ_i. Take $f \in \Delta$. Define $\varphi(t, x, f) : \mathbb{R} \times M \times \Delta \to M$ by

$$\varphi(t, x, f) = \varphi_t(x, f) = \phi_{i_m}(\tau, (\phi_{i_{m-1}}(h, ...\phi_{i_1}(h, x)))$$

where $\tau \equiv t \mod h$, and $f(s) = i_k$ for $s \in [(k-1)h, kh)$.

Note also that use of ϕ_i^{-1} defines this system backwards in time as well, and that ϕ, as a composition of continuous functions, remains continuous with respect to x.

Thus, $\varphi(t, x, f)$ is given by the flow along the dynamical system ϕ_i during the period of time for which $f = i$. With this, we can explicitly define our deterministic hybrid system. Consider

$$\Phi_t(x_0, f_0) \equiv \begin{pmatrix} \varphi_t(x_0, f_0) \\ \psi_t(f_0) \end{pmatrix} \tag{4.1}$$

with initial conditions $f_0 \in \Delta$, and $x_0 \in M$. Let $\psi_t(f_0) = f_t$ and notice that

$$\Phi_0(x_0, f_0) = \begin{pmatrix} x_0 \\ f_0 \end{pmatrix}$$

Then,

$$\Phi_{t+s}(x_0, f_0) = \begin{pmatrix} \varphi_{t+s}(x_0, f_0) \\ f_{t+s} \end{pmatrix} = \begin{pmatrix} \varphi_t(\varphi_s(x_0, f_0), f_s) \\ f_t \circ f_s \end{pmatrix} = \Phi_t \circ \Phi_s(x_0, f_0).$$

Thus, Φ_t is in fact a flow, so the deterministic hybrid system is a dynamical system.

This explanation is rather unintuitive and bulky. It is easier to consider a less rigorous definition of the system. Consider a point $(x, f) \in M \times \Delta$. We first consider the dynamical system that is given to us by $f(0)$. As time moves forward, the orbit of x is given by that dictated by the dynamical system corresponding to $\psi(f, t)(0)$. Recall that ψ simply shifts the function f to the right. f is piecewise constant; at time h, we may have that for all $\varepsilon > 0$, $\psi(f, h)(0) \neq \psi(f, h-\varepsilon)(0)$. If the function changes values, we then instantaneous switch which dynamical system is dictating the orbit of x. This could then result in a non-smooth orbit. This continues, with a possible change in dynamical system on M occurring after a time interval of length h. For the set $M \times \Delta$, we use the metric induced by the L_1 norm; that is, the metric in $M \times \Delta$ is given by the sum of the metrics used in M (usually given by the standard Euclidean norm in $\mathbb{R}^n$, and the metric used in Δ. Note then that by Tychonoff's Theorem that $M \times \Delta$ is compact, and that Φ is continuous (as it is continuous in each of its components separately).

It is helpful to have a terminology that explains the behavior in $M \times \Delta$, in that what happens on Δ is independent, but that the flow on M is dependent on what happens in Δ.

Definition 103. *A flow π on a product space $X \times Y$ is said to be a skew-product flow if there exist continuous mappings $\phi : X \times Y \times T \to X$ and $\sigma : (Y \times T \to Y)$ such that*

$$\pi(x, y, t) = (\psi(x, y, t), \sigma(y, t))$$

where σ itself is a flow on Y and T is the set of time values (for our purposes, we shall take $T = \mathbb{R}$.) (8)

It should not be too hard to see that our system is a skew-product flow, where X as in the definition above corresponds to our space M, and Y in the definition above corresponds to Δ.

Now that we have an understanding of the behavior of our system $(M \times \Delta, \Phi)$, we turn to examining some recurrence concepts. We begin by adapting our understanding of (ε, T)-chains from Chapter 2 to fit this new situation.

Definition 104. *A set $E \subset M$ is called a chain set of a system if (i) for all $x \in E$ there exists $f \in \Delta$ such that $\varphi(t, x, f) \in E$ for all $t \in \mathbb{R}$, (ii) for all $x, y \in E$ and for all $\varepsilon, T > 0$ there exist*

$n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $f_0, \ldots, f_{n-1} \in \Delta$ and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = y$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \ldots, n-1.$$

Such a sequence is called an (ε, T)-chain from x to y.

If there exists a (ε, T)-chain from x to y and from y to x, we say that x and y are chain equivalent.

Chain sets demonstrate some nice properties. It is not too difficult to show that chain sets are pairwise disjoint, compact, and connected, as given in the following lemmas.

Lemma 105. *Chain sets are pairwise disjoint.*

Proof. Let E_1 and E_2 be two chain sets, and let $x \in E_1 \cap E_2$ (that is, E_1 and E_2 are not disjoint). Then let $y \in E_1$ and $z \in E_2$. Then, given $\varepsilon, T > 0$, by definition of a chain set there exist $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $f_0, \ldots, f_{n-1} \in \Delta$ and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = y$, $x_n = x$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \ldots, n-1.$$

Similarly, for all $\varepsilon, T > 0$ there exist $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $f_0, \ldots, f_{n-1} \in \Delta$ and $t_0, \ldots, t_{n-1} \geq T$ with $x_0 = x$, $x_n = z$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \ldots, n-1.$$

Thus, the concatenation of these two ε, T-chains results in a ε, T-chain from y to z, and thus y and z are in the same chain set, and thus, since the choice of y and z was arbitrary, $E_1 = E_2$. □

Lemma 106. *Chain sets are compact.*

Proof. Let E be a chain set, and let $x \in M$ be a limit point of E. Then there exists a sequence in E, $\{x_i\}_{i=1}^{\infty}$ such that $x_i \to x$. Let $y \in E$, and let $\varepsilon, T > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $d(x_N, x) < \varepsilon$. By definition of a chain set, there exist $n \in \mathbb{N}$, $x_0, \ldots, x_n \in M$, $f_0, \ldots, f_n \in \Delta$ and $t_0, \ldots, t_n \geq T$ with $x_0 = y$, $\varphi(t_n, x_n, f_n) = x_N$, and

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) \leq \varepsilon \text{ for all } j = 0, \ldots, n-1.$$

Thus, since $d(\varphi(t_n, x_n, f_n), x) < \varepsilon$, setting $x_{n+1} = x$ gives an ε, T-chain from y to x, and similarly from x to y, and thus $x \in E$, and thus E is closed. Since E is then a closed subset of the compact set M, E is thus compact. □

Lemma 107. *Chain sets are connected.*

Proof. Let E be a chain set and let A, B be open sets such that $E \subset A \cup B$ and $A \cap B = \emptyset$. If $\inf\{d(a,b)|a \in A, b \in B\} > 0$, then there exists $\varepsilon < \inf\{d(a,b)|a \in A, b \in B\}$ and thus there exists no $\varepsilon, T-$chain from any $a \in A$ to any $b \in B$, and thus one of A or B must be empty. If $\inf\{d(a,b)|a \in A, b \in B\} = 0$, then there exists some x such that $\inf\{d(a,x)|a \in A\} = 0$ and $\inf\{d(b,x)|b \in B\} = 0$. Since by Lemma 106, E is closed, this implies that $x \in E$, and thus $x \in A$ or $x \in B$. Without loss of generality, let $x \in A$. Since $\inf\{d(b,x)|b \in B\} = 0$, this implies that if B is nonempty, for every neighborhood of N of X, $N \cap B \neq \emptyset$. However, since A is open, there is a neighborhood N of x such that $N \subset A$. This then implies that $A \cap B \neq \emptyset$, a contradiction. Thus B is empty, and E is connected.

□

Note that the above lemma does not show that chain sets are necessarily path connected; indeed, the following is an example of a chain set which is not path connected.

Example 108. *Let G, the graph governing Δ, be the complete graph on 2 vertices, and let $M = \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\} \cup \{0\} \times [-1, 1]$, otherwise known as the Topologist's Sine Curve. Let the two systems defined on M be given by the following differential equations:*

$$A : \dot{x} = -x(1/2\pi - x) \tag{4.2}$$

$$B : \dot{x} = x(1/2\pi - x) \tag{4.3}$$

(Note that it is sufficient to describe the dynamics of the systems with just the behavior in the x-coordinate alone as, except where $x = 0$ - which consists entirely of fixed points - there is exactly one y-value for each x-value.) Thus, in both systems, the set $\{0\} \times [-1, 1]$ is entirely made up of fixed points, and in System A, along the set $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ the system moves from right to left (with a fixed point at $x = 1/2pi$), the speed converging to zero

as x approaches zero. Similarly, in system B, along $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, the system moves left to right, with a fixed point at $x = 1/2\pi$. We then claim that the entirety of M is forms one chain set. Let $x, y \in M$. It should be clear that, if $x, y \in \{0\} \times [-1, 1]$, or if $x, y \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, then for all $\varepsilon, T > 0$, there exists an $\varepsilon, T-$chain from x to y. If $x \in \{0\} \times [-1, 1]$ and $y \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, then a chain can be formed by staying at x for a time of at least T, and then jumping by ε onto a point $z \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ (since M is the closure of $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, x is a limit point of $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$, and thus there exists a point z within ε of x). Thus, since we have established that there is a chain from z to y, there is a chain from x to y. Similarly, if $x \in \{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ and $y \in \{0\} \times [-1, 1]$, then, if we let z be a point on $\{(x, f(x))|x \in (0, 1/2\pi], f(x) = \sin(1/x)\}$ within ε of y, then there exists a chain from x to z, and then jumping to y gives a chain from x to y. Thus, M is a chain set.

It is well known that M, the Topologist's Sine Curve, is a set that is connected but not path connected. A proof can be found in (10), pages 137-138.

Thus, it is required that chain sets be connected, but they may not exhibit path connectivity.

Since a chain set E is a subset of M, it helps to have an extension of it to a set contained in $M \times \Delta$.

Definition 109. *Given $E \subset M$, the lift of E to $M \times \Delta$ is given by*

$$\ell(E) = \{(x, f) \in M \times \Delta, \Phi(t, x, f) \in E \text{ for all } t \in \mathbb{R}\}.$$

Definition 110. *A set $A \subset M$ is said to be invariant if for all $x \in A$, $\varphi(t, x, f) \in A$ for all $t \in \mathbb{R}$ and $f \in \Delta$. A set $A \subset M$ is said to be forward invariant if for all $x \in A$, $\varphi(t, x, f,) \in A$ for all $t \in \mathcal{R}^+$ and $f \in \Delta$. Similarly, a set $A \subset M$ is said to be backward invariant if for all $x \in A$, $\varphi(t, x, f,) \in A$ for all $t \in \mathcal{R}^-$ and $f \in \Delta$.*

Remark 111. *Notice that if E is invariant, $\ell(E) = \Delta \times E$.*

Since $(M \times \Delta, \Phi)$ is a dynamical system on a compact set, it has chain transitive sets. We would like to make connections between a chain transitive set contained in $M \times \Delta$ and a chain set that is a subset of M. The following results relate the ideas of chain transitive sets, chain sets, projections, and lifts, and what properties are retained when projecting onto M or lifting to $M \times \Delta$.

Theorem 112. *Let $\mathcal{E} \subset M \times \Delta$ be a maximal invariant chain transitive set for the flow. Then $\pi_M \mathcal{E}$ is a chain set.*

Proof. Let $\mathcal{E}$ be an invariant, chain transitive set in $M \times \Delta$. For $x \in \pi_M \mathcal{E}$ there exists $f \in \Delta$ such that $\varphi(t, x, f) \in \mathcal{E}$ for all t by invariance. Now let $x, y \in \pi_M \mathcal{E}$ and choose $\varepsilon, T > 0$. Then by chain transitivity of $\mathcal{E}$, we can choose x_j, f_j, t_j such that the corresponding trajectories satisfy the required condition. The proof is concluded by noticing that $\pi_M \mathcal{E}$ is maximal if $\mathcal{E}$ is maximal.

□

Lemma 113. *Given a maximal invariant chain transitive set $\mathcal{E} \subset M \times \Delta$, $\mathcal{E} \subset \ell(\pi_M \mathcal{E})$.*

Proof. Let $(x, f) \in \mathcal{E}$. Then $x \in \pi_M \mathcal{E}$. Since $\mathcal{E}$ is invariant, $\Phi(t, x, f) \in \mathcal{E}$ for all $t \in \mathbb{R}$. Thus, $\pi_M \Phi(t, x, f) = \varphi(t, x, f) \in \pi_M \mathcal{E}$ for all $t \in \mathbb{R}$. This then implies that $(x, f) \in \ell(\pi_M \mathcal{E})$, by definition of the lift. □

We then wondered if it was possible to establish a more general theory about chain sets and chain transitive sets, and how they are related via lifts and projections. In order to accomplish this task, we made use of the following theorem, taken from (2), Theorem 2.7.18.

Theorem 114. *If ϕ^t is a flow on a compact metric space (X, d) and $x, y \in X$, then the following statements are equivalent.*

1. *The points x and y are chain equivalent with respect to ϕ^t.*

2. *For every $\varepsilon > 0$ and $T > 0$ there exists an $(\varepsilon, 1)$-chain*

$$(x_0, \ldots, x_n; t_0, \ldots, t_{n-1})$$

from x to y such that

$$t_0 + \cdots + t_{n-1} \geq T,$$

and there exists an $(\varepsilon, 1)$*-chain*

$$(y_0, \ldots, y_m; s_0, \ldots, s_{m-1})$$

from y *to* x *such that*

$$s_0 + \cdots + s_{m-1} \geq T.$$

3. *For every* $\varepsilon > 0$ *there exists an* $(\varepsilon, 1)$*-chain from* x *to* y *and a* $(\varepsilon, 1)$*-chain from* y *to* x.

4. *The points* x *and* y *are chain equivalent with respect to* ϕ^1.

Notice then, that by this theorem, for chain sets E such that $E = \pi_M(\mathcal{E})$, where $\mathcal{E}$ is the lift of E, it is sufficient to take Definition 104 consider all chains where all t_i's take the value h (since the number 1 in part 4 of the above theorem is rather arbitrary). However, this may not be true for all chain sets in general, as there exist chain sets E such that $E \neq \pi_M(\mathcal{E})$.

Lemma 115. *If* G *is a complete graph, then given a maximal invariant chain transitive set* $\mathcal{E} \subset M \times \Delta$, $\mathcal{E} = \ell(\pi_M \mathcal{E})$.

Proof. It remains to show that $\ell(\pi_M \mathcal{E}) \subset \mathcal{E}$; this can be done by showing that $\ell(\pi_M \mathcal{E})$ is chain transitive.

Let $(x, f), (y, g) \in \mathcal{E}$ and pick $\varepsilon, T > 0$. Recall that

$$d(f, g) = \sum_{i=-\infty}^{\infty} \left(\frac{1}{h} \int_{ih}^{(i+1)h} \delta(f, g, t) dt \right) * 4^{-|i|}$$

where

$$\delta(f, g, t) = \begin{cases} 1 & f(t) \neq g(t) \\ 0 & f(t) = g(t) \end{cases}.$$

There exists $N \in \mathbb{N}$ such that

$$2 \sum_{i=N}^{\infty} \frac{1}{4^{|i|}} < \varepsilon/2.$$

Chain transitivity of $\mathcal{E}$ means there exists $k \in \mathbb{N}$ and $x_0, \ldots, x_k \in M$, $f_0, \ldots, f_{k-1} \in \overline{\Delta}$, $t_0, \ldots, t_{k-1} > T$ with $x_0 = \varphi(2T, x, f)$ and $x_k = \varphi(-T, y, g)$ with

$$d(\varphi(t_j, x_j, f_j), x_{j+1}) < \varepsilon.$$

Without loss of generality, let $t > 1$. Then by Theorem 114, we can set

$$t_0 = \cdots = t_{k-1} = h.$$

Define

$$t_{-2} = Nh,\;\; x_{-2} = x,\;\; g_{-2} = f$$

$$t_{-1} = t, x_{-1} = \varphi(Nh, x, f), g_{-1} = \begin{cases} f(t_{-2} + t) & t \leq t_1 \\ f_0(t - t_{-1}) & t > t_1 \end{cases}$$

Let $t_0, \;\ldots,\; t_{k-1}$ and $x_0, \ldots, x_k$ be given as before, and let

$$t_k = Nh, x_{k+1} = y, g_{k+1} = g.$$

Now, for $j = 0, \ldots, k-2$ we define

$$g_j(t) = \begin{cases} g_{j-1}(t_{j-1} + t) & t \leq 0 \\ f_j(t) & 0 < t \leq t_j \\ f_{j+1}(t - t_j) & t > t_j \end{cases}$$

$$g_{k-1} = \begin{cases} g_{k-2}(t_{k-2} + t) & t \leq 0 \\ f_{k-1}(t) & 0 < t \leq t_{k-1} \\ g(t - t_{k-1} - Nh) & t > t_{k-1} \end{cases}$$

$$g_k = \begin{cases} g_{k-1}(t_{k-1} + t) & t \leq 0 \\ g(t - Nh) & t > 0 \end{cases}$$

We then claim that, by construction, all g_j's are elements of $\overline{\Delta}$. Recall firstly that by Definition 77 functions in $\overline{\Delta}$ require that

$$\{f(ih)\}_{i \in \mathbb{Z}} \in \Omega$$

for all $f \in \overline{\Delta}$. Since the graph G is complete, clearly for each f_i the jumps between vertices are allowed by the graph. The "stitching" together of pieces of the functions f_i's is also allowed by the graph G associated $\overline{\Delta}$ because of the completeness of G.

We further require that the functions g_i be piecewise constant on intervals of length h. By setting $t_j = Nh$ for all $j \in \{-2, -1, \ldots, k-1\}$ this property is satisfied as well. Thus, $g_j \in \overline{\Delta}$

for all j.

We further claim that for all $j = -2, -1, \ldots, k$,

$$d(g_j(t_j + \cdot), g_{j+1}) < \varepsilon.$$

By choice of N, on has that for all $d_1, d_2 \in \overline{\Delta}$

$$\begin{aligned} d(d_1, d_2) &= \sum_{i=-\infty}^{\infty} \left(\frac{1}{h} \int_{ih}^{(i+1)h} \delta(d_1, d_2, t) dt \right) * 4^{-|i|} \\ &\leq \sum_{i=-N}^{N} \left\{ \left(\frac{1}{h} \int_{ih}^{(i+1)h} \delta(d_1, d_2, t) dt \right) * 4^{-|i|} \right\} + \varepsilon/2 \end{aligned}$$

Thus it suffices to show that for the considered functions, the integrands vanish. Notice by definition, for all $i \in \{-2, -1, \ldots, k-1\}$, $g_i(t + Nh) = g_{i+1}$ for all $-Nh < t < Nh$. Thus, $\delta(g_i(t + Nh), g_{i+1}(t), t) = 0$ for all $-Nh < t < Nh$, and therefore,

$$\int_{ih}^{(i+1)h} \delta(f, g, t) dt$$

for all $i \in \{-N, \ldots, N-1\}$. Thus for all $j = -2, -1, \ldots, k$,

$$d(g_j(t_j + \cdot), d_{j+1}) < \varepsilon.$$

□

It is important to remember that chain sets are not chain transitive sets as defined in Definition 62. This is because we can not consider the behavior on M alone as a flow; it is dependent on orbits in Δ. The following example shows why these concepts are not the same.

Example 116. *Consider the system where Δ is given by the complete graph on two vertices, and $M = [0, 2]$. Let the system corresponding to vertex A be given by the differential equation:*

$$\dot{x} = -x(x-1)(x-2),$$

and the system corresponding to vertex B is given by

$$\dot{x} = -x(x-2).$$

Both of these systems are bounded on either end by fixed points at 0 and 2. System A has a repelling fixed point at 1*, while in system* B *on the interval* $(0,2)$ *the flow moves in the positive direction. We claim that in this system, the interval* $[0,1]$ *is a chain set for all* $T, \varepsilon > 0$*. (The fixed point at* $x = 2$ *is also rather trivially a chain set.) Note that on the open interval* $(0,1)$*, the flow moves in two different directions in each system. Note further that the lift of* $[0,1]$ *is not equal to* $\Delta \times [0,1]$*, as* $[0,1]$ *is not invariant. Then let* $y, z \in [0,1)$*. Given* $\varepsilon, T > 0$*, we wish to construct an* $(\varepsilon, T)-$*chain from* y *to* z*. Consider the sequence* $a_n = \varphi(-nT, z, B)$*. Notice then that*

$$\lim_{n\to\infty} a_n = 0.$$

Let N *be such that* $|0 - a_N| < \varepsilon/2$*. Notice that there exists a time* $T' > T$ *such that*

$$|0 - \varphi(T', y, A)| < \varepsilon/2.$$

Thus, by the triangle inequality,

$$|\varphi(T', y, A) - a_N| < \varepsilon.$$

Therefor, the sequence $y, \varphi(T', y, A), a_N, z$ *forms an* $(\varepsilon, T)-$*chain from* y *to* z*. Since we know chain sets are closed by Lemma 106, we know now that* $[0,1]$ *is a chain set for all* $T, \varepsilon > 0$*.*

Now, in the above system, let the graph G *be the cycle on two vertices. Then we claim that* $[0,1]$ *is no longer a chain set for all* $T > 0$*. Note now that the only functions in* Δ *are shifts of the periodic function that switches between* A *and* B *on intervals of length* h*. Without loss of generality, let* $z < y$ *such that* $\varphi(h, \varphi(h, z, A), B) \neq z$*. Such a point exists because for all* $\varepsilon > 0$*, there exists a point* $x \in (1/2, 1)$ *such that* $\varphi'(t, x, A) < \varepsilon$ *as the flow in system* A *converges to 1. However, since* $\phi'(t, 1, B) \neq 0$*, the flow is not symmetric, and thus we can not have that* $\varphi(h, \varphi(h, z, A), B) = z$ *for all* $z \in (0,)$*, and thus such a point* z *exists. We would like to show that there exist* ε, T *such that there is no longer an* (ε, T)*-chain from* z *to* y*. Let us start at* z *with system* A*. Let* $T = 2h$*, and* $x_{2h} = \varphi(h, \varphi(h, z, A), B)$*. If* $x_{2h} < z$*, then pick* $\varepsilon < |z - x_{2h}|$ *- notice now that all solutions must be less than* z *for all times* $t > h$*. Similarly, if* $x_{2h} > z$*, we can choose* ε *such that we can not reach any points less than* z *with a particular* ε*. If* $z = x_{2h}$*, let* ε *be small enough such that* $\varphi(-h, 1 + \varepsilon, B) < z$*.*

However, note that if we take $T = h$, since we are allowed to switch functions after each ε-jump, we are essentially in the same case as when the graph G is complete (since we may jump by $\varepsilon = 0$ and let $f(0)$ take either value A or B after the jump). Thus, in this case $[0, 1]$ is an (ε, h)-chain set. Note that this example then implies that, (ε, h)-chain sets and (ε, T)-chain sets for a general T may not be equivalent, and thus Theorem 114 does not apply in this case.

The above example illustrates that ε, h-chain sets are like having the complete graph (see Lemma 115); that is, if $\mathcal{E} \subset \Delta \times M$ is a maximal invariant chain transitive set, then $\mathcal{E} = \ell(\pi_M \mathcal{E})$, and $\pi_M \mathcal{E}$ is a chain set for all T (see Theorem 112).

Thus, we see that the relationship between chain sets in M and chain transitive sets in $M \times \Delta$ is rather complicated. As of right now, there is no general theory about the relationship between the two concepts. Above we have explored certain examples of relationships, but future work may entail coming up with a more general result that relates the two. In addition, concepts such as Poincaré recurrence and nonwandering sets could be explored within this context.

CHAPTER 5. CONCLUSION

To conclude, in this work we have examined two main topics: the symbolic dynamics that exist on Ω given a finite N-graph G, and then the concept of a hybrid dynamical system on $M \times \Delta$. In the section on symbolic dynamic, we introduced a space Ω that consists of bi-infinite sequences whose entries consist of vertices of G, ordered such that they are essentially bi-infinite paths of the graph. We defined a metric on Ω that induces a topology such that entries closer to the origin point contribute more to the distance between two sequences than entries further out in the tails. It was demonstrated that under this topology, Ω is compact and thus α- and ω-limit sets are nonempty for every $x \in \Omega$, and characterized these $\alpha-$ and ω-limit sets, as well as different types of recurrent sets, such as Poincaré sets, nonwandering sets, and chain recurrent sets. We finally were able to show that the lifts of communicating classes form a Morse decomposition for Ω, and that the behavior within them is topologically transitive and sensitively dependent on initial chaos, thus demonstrating chaotic behavior. Furthermore, because of the topological transitivity on the lifts of communicating classes, these sets form a finest Morse decomposition on Ω. Further questions that could be examined could include what happens if the graph G is allowed to have infinitely many vertices, in which case Ω is probably not compact.

In the next section, we discussed the concept of switching between a finite number of flows $\phi_1, \phi_2, \ldots, \phi_k$ on the same compact space $M \subset \mathbb{R}^n$, where the switching is determined by the above N-graph G. We thus adapted Ω into a continuous time dynamical system Δ, in order to consider the continuous time skew product flow $M \times \Delta$. We showed that Δ possessed essentially the same characteristics as Ω: it is still compact, the lifts of communicating classes form a finest Morse decomposition, and within the lifts there is topological transitivity and sensitive dependence on initial conditions (and thus chaos). Thus, Δ on its own is wholly understood. It is then embedded in the skew product flow Φ on $M \times \Delta$, where the flow on Δ is given by the right time shift ψ and the "flow" (because of the dependence on Δ, the behavior on M alone can not be considered a true flow or dynamical system) on M is given by the switching between the systems $\phi_1, \phi_2, \ldots, \phi_k$ on M and dictated by a function in Δ. We then introduced the concept of a "chain set" which adapts the idea of chain transitive sets on a flow for the behavior on M and examined the relationship of chain sets to chain transitive sets that are subsets of the higher space $M \times \Delta$. As we discovered, it is not so easy to relate the two sets through projections and lifts. In the case of the complete graph, chain sets are exactly the projections of chain transitive sets, but this may not be a general theory. Thus, further work could include formulating a more general theory about the relationship between chain sets and chain transitive sets via lifts and projections. In addition, since by Tychonoff's theorem $M \times \Delta$ is compact under the product topology, we are guaranteed the existence of non empty α, ω-limit sets. So future work may involve characterizing these limit sets and perhaps identifying a non-trivial Morse decomposition for the dynamical system. Other recurrence concepts aside from chain recurrence could be explored as well, such as the identification of Poincaré sets and nonwandering sets.

BIBLIOGRAPHY

[1] J. Ackerman, K. Ayers, E. J. Beltran, J. Bonet, D. Lu, T. Rudelius, "A Behavioral Characterization of Discrete Time Dynamical Systems using Directed Graphs". Iowa State University, Ames, 2011.

[2] J. Alongi, G. Nelson, *Recurrence and Topology*, American Mathematical Society, 2007.

[3] J. Ayala-Hoffmann, P. Corbin, K, McConville, F. Colonius, W. Kliemann, J. Peters, "Morse Decompositions, Attractors and Chain Recurrence". Iowa State University, Ames 2007.

[4] F. Colonius and W. Kliemann, *The Dynamics of Control*, Birkhäuser, 2000.

[5] X. Garcia, J. Kunze, T. Rudelius, A. Sanchez, S. Shao, E. Speranza, C.Vidden, "Invariant Measures for Hybrid Stochastic Systems". Iowa State University, Ames, 2011.

[6] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[7] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1998.

[8] R. Sacker and G. Sell, "Skew-Product Flows, Finite Extensions of Minimal Transformation Groups and Almost Periodic Differential Equations", *Bulletin of the American Mathematical Society*, Volume 79, Number 4, July 1973, 702-805.

[9] S. Smale, "Finding a Horseshoe on the Beaches of Rio", *The Mathematical Intelligencer*, Volume 20, Issue 1, March 1998, pp 39-44.

[10] L. Arthur Steen and J. Arthur Seebach, *Counterexamples in Topology*, Dover Publications, 1978.